INTUITIONISM

AN INTRODUCTION

A. HEYTING

Emeritus Professor of Mathematics
University of Amsterdam



1971

NORTH-HOLLAND PUBLISHING COMPANY – AMSTERDAM · LONDON

©North-Holland Publishing Company, Amsterdam, 1956

All Rights Reserved. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photo copying, recording or otherwise, without the prior permission of the Copyright owner.

First edition 1956
Second revised edition 1966
Third edition 1971

ISBN 7204 2239 6

PRINTED IN THE NETHERLANDS

PREFACE

In order to prevent the reader from wasting his time in useless attempts to solve supposed riddles, I warn him that the persons of the dialogue are not caricatures of living or deceased persons, much less their doubles. They are pegs to hang ideas on, and nothing else. To a certain extent this is even true for Int, who represents the position of intuitionism. For the sake of clearness I made him speak sometimes in a somewhat more absolute way than I should have done if I had freely expressed my own opinions. The discussion is strictly limited to intuitionism; other conceptions of mathematics are touched on only in so far as they lead to objections against intuitionism. I reject any reproach for incomplete exposition of other points of view.

It was necessary to give proofs in great detail, even where they differ only by small additions from the well-known classical ones. There was no other way to indicate in which places these additions had to be made. In the course of the book, as the reader is supposed to develop a feeling for the specifically intuitionistic difficulties, I have gradually adopted a more condensed style.

I thank all those have contributed to improve the book, among them Dr. Paul Gilmore, Prof. Leon Henkin and Mr. William Tait, who read parts of the manuscript and suggested many linguistic improvements, Mr. J. J. de Iongh and Mr. F. van de Oudeweetering, who accurately revised the manuscript. Mr. de Iongh indicated many corrections and clarifications in the text.

In many places of the book the reader will find old-fashioned reasonings which lack generality and which are more clumsy than the modern methods. This has different reasons. In the first place, the powerful methods often make an excessive use of indirect proof, so that it is almost impossible to introduce them in intuitionistic mathematics. In the second place, the very general modern theories proceed by the axiomatic method. Now this method can only work well, if some concrete theories exist, from which the axiomatic theory can be constructed by generalization

For instance, general topology could only be developed after the topology of euclidean spaces was known in some detail. As a matter of fact, almost no part of intuitionistic mathematics has been investigated deeply enough to admit the construction of a general axiomatic theory. Thus in this book I had to confine myself to the most elementary case of integration; when this will be better known than it is at present, it will become possible to construct an axiomatic theory on the subject. Even in the case of algebra, where axiomatization is possible at this moment, it seemed better to treat the concrete example of the real number field, in view of the fact that the book is meant as an introduction.

Probably in some cases I used antiquated methods because I did not know the modern ones. One of the aims of the book is, to enable working mathematicians to decide, which of their results can be proved intuitionistically. Intuitionism can only flourish, if mathematicians, working in different fields, become actively interested in it and make contributions to it. In order to build up a definite branch of intuitionistic mathematics, it is necessary in the first place to have a thorough knowledge of the corresponding branch of classical mathematics, and in the second place to know by experience where the intuitionistic pitfalls lie. I try in this book to teach the latter; I hope that some of my readers will give a more satisfactory treatment of details than I could, or that they will treat other theories intuitionistically. The "reading suggestions" are intended to help them; they indicate the most important intuitionistic work on some special subjects.

References have been made in the following way.
"Th. 2" refers to theorem 2 of the same section.
"Def. 2" refers to definition 2 of the same section.
"6.2.1, Th. 2" refers to theorem 2 of section 6.2.1.

A. HEYTING

Amsterdam, August 1955.

PREFACE TO THE SECOND EDITION

In this reprint I have made only minor changes and additions. Only section 6.5 has been partly rewritten. It would have been tempting to add chapters on the relations of intuitionism to the theory of recursive functions and to the recent Russian investigations on constructive analysis; however, it seemed better to confine the content to one subject.

Amsterdam, March 1965. A. HEYTING

PREFACE TO THE THIRD EDITION

During the last few years the study of intuitionism has received new impulses from work on the theory of choice sequences. I have not treated this subject in detail, because it is still in a state of development, but I have mentioned the most important literature in section 3.4.2. Further I have rewritten section 3.4, using a manuscript by A. S. Troelstra, which he wrote for this purpose. I am most grateful to him for doing this. Many smaller corrections were suggested by the notes which A. A. Markov added to the Russian translation of the first edition.

Castricum, June 1970. A. HEYTING

Persons of the dialogue: Class, Form, Int, Letter, Prag, Sign.

I

DISPUTATION

CLASS. How do you do, Mr. Int? Did you not flee the town on this fine summer day?

INT. I had some ideas and worked them out at the library.

CLASS. Industrious bee! How are you getting along?

INT. Quite well. Shall we have a drink?

CLASS. Thank you. I bet you worked on that hobby of yours, rejection of the excluded middle, and the rest. I never understood why logic should be reliable everywhere else, but not in mathematics.

INT. We have spoken about that subject before. The idea that for the description of some kinds of objects another logic may be more adequate than the customary one has sometimes been discussed. But it was Brouwer who first discovered an object which actually requires a different form of logic, namely the mental mathematical construction [L. E. J. Brouwer 1908]. The reason is that in mathematics from the very beginning we deal with the infinite, whereas ordinary logic is made for reasoning about finite collections.

CLASS. I know, but in my eyes logic is universal and applies to the infinite as well as to the finite.

INT. You ought to consider what Brouwer's program was [L. E. J. Brouwer 1907]. It consisted in the investigation of mental mathematical construction as such, without reference to questions regarding the nature of the constructed objects, such as whether these objects exist independently of our knowledge of them. That this point of view leads immediately to the rejection of the principle of excluded middle, I can best demonstrate by an example.

Let us compare two definitions of natural numbers, say k and l.

I. k is the greatest prime such that $k-1$ is also a prime, or $k=1$ if such a number does not exist.
II. l is the greatest prime such that $l-2$ is also a prime, or $l=1$ if such a number does not exist.
Classical mathematics neglects altogether the obvious difference in character between these definitions. k can actually be calculated ($k=3$), whereas we possess no method for calculating l, as it is not known whether the sequence of pairs of twin primes p, $p+2$ is finite or not. Therefore intuitionists reject II as a definition of an integer; they consider an integer to be well defined only if a method for calculating it is given. Now this line of thought leads to the rejection of the principle of excluded middle, for if the sequence of twin primes were either finite or not finite, II would define an integer.
CLASS. One may object that the extent of our knowledge about the existence or non-existence of a last pair of twin primes is purely contingent and entirely irrelevant in questions of mathematical truth. Either an infinity of such pairs exist, in which case $l=1$; or their number is finite, in which case l equals the greatest prime such that $l-2$ is also a prime. In every conceivable case l is defined; what does it matter whether or not we can actually calculate the number?
INT. Your argument is metaphysical in nature. If "to exist" does not mean "to be constructed", it must have some metaphysical meaning. It cannot be the task of mathematics to investigate this meaning or to decide whether it is tenable or not. We have no objection against a mathematician privately admitting any metaphysical theory he likes, but Brouwer's program entails that we study mathematics as something simpler, more immediate than metaphysics. In the study of mental mathematical constructions "to exist" must be synonymous with "to be constructed".
CLASS. That is to say, as long as we do not know if there exists a last pair of twin primes, II is not a definition of an integer, but as soon as this problem is solved, it suddenly becomes such a definition. Suppose on January 1, 1970 it is proved that an infinity of twin primes exists; from that moment $l=1$. Was $l=1$ before that date or not? [Menger 1930].

INT. A mathematical assertion affirms the fact that a certain mathematical construction has been effected. It is clear that before the construction was made, it had not been made. Applying this remark to your example, we see that before Jan. 1, 1970 it had not been proved that $l=1$. But this is not what you mean. It seems to me that in order to clarify the sense of your question you must again refer to metaphysical concepts: to some world of mathematical things existing independently of our knowledge, where "$l=1$" is true in some absolute sense. But I repeat that mathematics ought not to depend upon such notions as these. In fact all mathematicians and even intuitionists are convinced that in some sense mathematics bear upon eternal truths, but when trying to define precisely this sense, one gets entangled in a maze of metaphysical difficulties. The only way to avoid them is to banish them from mathematics. This is what I meant by saying that we study mathematical constructions as such and that for this study classical logic is inadequate.

CLASS. Here come our friends Form and Letter. Boys, we are having a most interesting discussion on intuitionism.

LETTER. Could you speak about anything else with good old Int? He is completely submerged in it.

INT. Once you have been struck with the beauty of a subject, devote your life to it!

FORM. Quite so! Only I wonder how there can be beauty in so indefinite a thing as intuitionism. None of your terms are well-defined, nor do you give exact rules of derivation. Thus one for ever remains in doubt as to which reasonings are correct and which are not [R. Carnap 1934, p. 41; 1937, p. 46] [W. Dubislav 1932, p. 57, 75]. In daily speech no word has a perfectly fixed meaning; there is always some amount of free play, the greater, the more abstract the notion is. This makes people miss each other's point, also in non-formalized mathematical reasonings. The only way to achieve absolute rigour is to abstract all meaning from the mathematical statements and to consider them for their own sake, as sequences of signs, neglecting the sense they may convey. Then it is possible to formulate definite rules for deducing new statements from those already known and to avoid the uncertainty resulting from the ambiguity of language.

INT. I see the difference between formalists and intuitionists mainly as one of taste. You also use meaningful reasoning in what Hilbert called metamathematics, but your purpose is to separate these reasonings from purely formal mathematics and to confine yourself to the most simple reasonings possible. We, on the contrary, are interested not in the formal side of mathematics, but exactly in that type of reasoning which appears in metamathematics; we try to develop it to its farthest consequences. This preference arises from the conviction that we find here one of the most fundamental faculties of the human mind.

FORM. If you will not quarrel with formalism, neither will I with intuitionism. Formalists are among the most pacific of mankind. Any theory may be formalized and then becomes subject to our methods. Also intuitionistic mathematics may and will be thus treated [R. Carnap 1934, p. 44; 1937, p. 51].

CLASS. That is to say, intuitionistic mathematics ought to be studied as a part of mathematics. In mathematics we investigate the consequences of given assumptions; the intuitionistic assumptions may be interesting, but they have no right to a monopoly.

INT. Nor do we claim that; we are content if you admit the good right of our conception. But I must protest against the assertion that intuitionism starts from definite, more or less arbitrary assumptions. Its subject, constructive mathematical thought, determines uniquely its premises and places it beside, not interior to classical mathematics, which studies another subject, whatever subject that may be. For this reason an agreement between formalism and intuitionism by means of the formalization of intuitionistic mathematics is also impossible. It is true that even in intuitionistic mathematics the finished part of a theory may be formalized. It will be useful to reflect for a moment upon the meaning of such a formalization. We may consider the formal system as the linguistic expression, in a particularly suitable language, of mathematical thought.

If we adopt this point of view, we clash against the obstacle of the fundamental ambiguousness of language. As the meaning of a word can never be fixed precisely enough to exclude every possibility of misunderstanding, we can never be mathematically sure that the formal system expresses correctly our mathematical thoughts.

However, let us take another point of view. We may consider the formal system itself as an extremely simple mathematical structure; its entities (the signs of the system) are associated with other, often very complicated, mathematical structures. In this way formalizations may be carried out inside mathematics, and it becomes a powerful mathematical tool. Of course, one is never sure that the formal system represents fully any domain of mathematical thought; at any moment the discovering of new methods of reasoning may force us to extend the formal system.

FORM. For several years we have been familiar with this situation. Gödel's incompleteness theorem showed us that any consistent formal system of number-theory may be extended consistently in different ways.

INT. The difference is that intuitionism proceeds independently of the formalization, which can but follow after the mathematical construction.

CLASS. What puzzles me most is that you both seem to start from nothing at all. You seem to be building castles in the air. How can you know if your reasoning is sound if you do not have at your disposal the infallible criterion given by logic? Yesterday I talked with Sign, who is still more of a relativist than either of you. He is so slippery that no argument gets hold of him, and he never comes to any somewhat solid conclusion. I fear this fate for anybody who discards the support of logic, that is, of common sense.

SIGN. Speak of the devil and his imp appears. Were you speaking ill of me?

CLASS. I alluded to yesterday's discussion. To-day I am attacking these other two damned relativists.

SIGN. I should like to join you in that job, but first let us hear the reply of your opponents. Please meet my friend Prag; he will be interested in the discussion.

FORM. How do you do? Are you also a philosopher of science?

PRAG. I hate metaphysics.

INT. Welcome, brother!

FORM. Why, I would rather not defend my own position at the moment, as our discussion has dealt mainly with intuitionism and we might easily confuse it. But I fear that you are wrong as to

intuitionistic logic. It has indeed been formalized and valuable work in this field has been done by a score of authors. This seems to prove that intuitionists esteem logic more highly than you think, though it is another logic than you are accustomed to. INT. I regret to disappoint you. Logic is not the ground upon which I stand. How could it be? It would in turn need a foundation, which would involve principles much more intricate and less direct than those of mathematics itself. A mathematical construction ought to be so immediate to the mind and its result so clear that it needs no foundation whatsoever. One may very well know whether a reasoning is sound without using any logic; a clear scientific conscience suffices. Yet it is true that intuitionistic logic has been developed. To indicate what its significance is, let me give you an illustration. Let A designate the property of an integer of being divisible by 8, B the same by 4, C the same by 2. For $8a$ we may write $4 \times 2a$; by this mathematical construction P we see that the property A entails B ($A \rightarrow B$). A similar construction Q shows $B \rightarrow C$. By effecting first P, then Q (juxtaposition of P and Q) we obtain $8a = 2 \times (2 \times 2a)$ showing $A \rightarrow C$. This process remains valid if for A, B, C we substitute arbitrary properties: If the construction P shows that $A \rightarrow B$ and Q shows that $B \rightarrow C$, then the juxtaposition of P and Q shows that $A \rightarrow C$. We have obtained a logical theorem. The process by which it is deduced shows us that it does not differ essentially from mathematical theorems; it is only more general, e.g. in the same sense that "addition of integers is commutative" is a more general statement than "$2+3=3+2$". This is the case for every logical theorem: it is but a mathematical theorem of extreme generality; that is to say, logic is a part of mathematics, and can by no means serve as a foundation for it. At least, this is the conception of logic to which I am naturally led; it may be possible and desirable to develop other forms of logic for other purposes.

It is the mathematical logic which I just described that has been formalized. The resulting formal system proves to have peculiar properties, very interesting when compared to those of other systems of formal logic. This fact has led to the investigations to which Mr. Form alluded, but, however interesting, they are tied but very loosely to intuitionistic mathematics.

LETTER. In my opinion all these difficulties are imaginary or artificial. Mathematics is quite a simple thing. I define some signs and I give some rules for combining them; that is all.

FORM. You want some modes of reasoning to prove the consistency of your formal system.

LETTER. Why should I want to prove it? You must not forget that our formal systems are constructed with the aim towards applications and that in general they prove useful; this fact would be difficult to explain if every formula were deducible in them. Thereby we get a practical conviction of consistency which suffices for our work. What I contest in intuitionism is the opinion that mathematics has anything to do with the infinite. I can write down a sign, say α, and call it the cardinal number of the integers. After that I can fix rules for its manipulation in agreement with those which Mr. Class uses for this notion; but in doing this I operate entirely in the finite. As soon as the notion of infinity plays a part, obscurity and confusion penetrate into the reasoning. Thus all the intuitionistic assertions about the infinite seem to me highly ambiguous, and it is even questionable whether such a sign as $10^{10^{10}}$ has any other meaning than as a figure on paper with which we operate according to certain rules [J. Dieudonné 1949].

INT. Of course your extreme finitism grants the maximum of security against misunderstanding, but in our eyes it implies a denial of understanding which it is difficult to accept. Children in the elementary school understand what the natural numbers are and they accept the fact that the sequence of natural numbers can be indefinitely continued.

LETTER. It is suggested to them that they understand.

INT. That is no objection, for every communication by means of language may be interpreted as suggestion. Also Euclid in the 20th proposition of Book IX, where he proved that the set of prime numbers is infinite, knew what he spoke about. This elementary notion of natural numbers, familiar to every thinking creature, is fundamental in intuitionistic mathematics. We do not claim for it any form of certainty or definiteness in an absolute sense, which would be unrealizable, but we contend that it is sufficiently clear to build mathematics upon.

LETTER. My objection is that you do not suppose too little, as

Mr. Class thinks, but far too much. You start from certain principles which you take as intuitively clear without any explanation and you reject other modes of reasoning without giving any grounds for that discrimination. For instance, to most people the principle of the excluded middle seems at least as evident as that of complete induction. Why do you reject the former and accept the latter? Such an unmotivated choice of first principles gives to your system a strongly dogmatic character.

INT. Indeed intuitionistic assertions must seem dogmatic to those who read them as assertions about facts, but they are not meant in this sense. Intuitionistic mathematics consists, as I have explained already to Mr. Class, in mental constructions; a mathematical theorem expresses a purely empirical fact, namely the success of a certain construction. "$2+2=3+1$" must be read as an abbreviation for the statement: "I have effected the mental constructions indicated by "$2+2$" and by "$3+1$" and I have found that they lead to the same result." Now tell me where the dogmatic element can come in; not in the mental construction itself, as is clear by its very nature as an activity, but no more in the statements made about the constructions, for they express purely empirical results.

LETTER. Yet you contend that these mental constructions lead to some sort of truth; they are not a game of solitaire, but in some sense must be of value for mankind, or you would be wrong in annoying others with them. It is in this pretence that I see the dogmatic element. The mathematical intuition inspires you with objective and eternal truths; in this sense your point of view is not only dogmatic, but even theological [H. B. Curry 1951, p. 6].

INT. In the first instance, my mathematical thoughts belong to my individual intellectual life and are confined to my personal mind, as is the case for other thoughts as well. We are generally convinced that other people have thoughts analogous to our own and that they can understand us when we express our thoughts in words, but we also know that we are never quite sure of being faultlessly understood. In this respect, mathematics does not essentially differ from other subjects; if for this reason you consider mathematics to be dogmatic, you ought to call any human reasoning dogmatic. The characteristic of mathematical thought

is, that it does not convey truth about the external world, but is only concerned with mental constructions. Now we must distinguish between the simple practice of mathematics and its valuation. In order to construct mathematical theories no philosophical preliminaries are needed, but the value we attribute to this activity will depend upon our philosophical ideas.

SIGN. In the way you treat language you put the clock back. Primitive language has this floating, unsteady character you describe, and the language of daily life is still in the main of the same sort, but as soon as scientific thought begins, the formalization of language sets in. In the last decades significists have studied this process. It has not yet come to an end, for more strictly formalized languages are still being formed.

INT. If really the formalization of language is the trend of science, then intuitionistic mathematics does not belong to science in this sense of the word. It is rather a phenomenon of life, a natural activity of man, which itself is open to study by scientific methods; it has actually been studied by such methods, namely that of formalizing intuitionistic reasoning and the signific method, but it is obvious that this study does not belong to intuitionistic mathematics, nor do its results. That such a scientific examination of intuitionistic mathematics will never produce a complete and definite description of it, no more than a complete theory of other phenomena is attainable, is clearly to be seen. Helpful and interesting as these meta-intuitionistic considerations may be, they cannot be incorporated into intuitionistic mathematics itself. Of course, these remarks do not apply to formalization inside mathematics, as I described it a few moments ago.

PRAG. Allow me to underline what Mr. Sign said just now. Science proceeds by formalization of language; it uses this method because it is efficient. In particular the modern completely formalized languages have appeared to be most useful. The ideal of the modern scientist is to prepare an arsenal of formal systems ready for use from which he can choose, for any theory, that system which correctly represents the experimental results. Formal systems ought to be judged by this criterion of usefulness and not by a vague and arbitrary interpretation, which is preferred for dogmatic or metaphysical reasons.

INT. It seems quite reasonable to judge a mathematical system by its usefulness. I admit that from this point of view intuitionism has as yet little chance of being accepted, for it would be premature to stress the few weak indications that it might be of some use in physics [J. L. Destouches 1951]; in my eyes its chances of being useful for philosophy, history and the social sciences are better. In fact, mathematics, from the intuitionistic point of view, is a study of certain functions of the human mind, and as such it is akin to these sciences. But is usefulness really the only measure of value? It is easy to mention a score of valuable activities which in no way support science, such as the arts, sports, and light entertainment. We claim for intuitionism a value of this sort, which it is difficult to define beforehand, but which is clearly felt in dealing with the matter. You know how philosophers struggle with the problem of defining the concept of value in art; yet every educated person feels this value. The case is analogous for the value of intuitionistic mathematics.

FORM. For most mathematicians this value is affected fatally by the fact that you destroy the most precious mathematical results; a valuable method for the foundation of mathematics ought to save as much as possible of its results [D. Hilbert 1922]. This might even succeed by constructive methods; for definitions of constructiveness other than that advocated by the intuitionists are conceivable. For that matter, even the small number of actual intuitionists do not completely agree about the delimination of the constructive. The most striking example is the rejection by Griss of the notion of negation, which other intuitionists accept as perfectly clear [H. Freudenthal 1936A] [G. F. C. Griss 1946, p. 24; 1946A]. It seems probable, on the other hand, that a somewhat more liberal conception of the constructive might lead to the saving of the vital parts of classical mathematics.

INT. As intuitionists speak a non-formalized language, slight divergences of opinion between them can be expected. Though they have arisen sooner and in more acute forms than we could foresee, they are in no way alarming, for they all concern minor points and do not affect the fundamental ideas, about which there is complete agreement. Thus it is most unlikely that a wider conception of constructiveness could obtain the support of in-

tuitionists. As to the mutilation of mathematics of which you accuse me, it must be taken as an inevitable consequence of our standpoint. It can also be seen as the excision of noxious ornaments, beautiful in form, but hollow in substance, and it is at least partly compensated for by the charm of subtle distinctions and witty methods by which intuitionists have enriched mathematical thought.

FORM. Our discussion has assumed the form of a discussion of values. I gather from your words that you are ready to acknowledge the value of other conceptions of mathematics, but that you claim for your conception a value of its own. Is that right?

INT. Indeed, the only positive contention in the foundation of mathematics which I oppose is that classical mathematics has a clear sense; I must confess that I do not understand that. But even those who maintain that they do understand it might still be able to grasp our point of view and to value our work.

LETTER. It is shown by the paradoxes that classical mathematics is not perfectly clear.

FORM. Yes, but intuitionistic criticism goes much farther than is necessary to avoid the paradoxes; Mr. Int has not even mentioned them as an argument for his conception, and no doubt in his eyes consistency is but a welcome by-product of intuitionism.

SIGN. You describe your activity as mental construction, Mr. Int, but mental processes are only observable through the acts to which they lead, in your case through the words you speak and the formulas you write. Does not this mean that the only way to study intuitionism is to study the formal system which it constructs?

INT. When looking at the tree over there, I am convinced I see a tree, and it costs considerable training to replace this conviction by the knowledge that in reality lightwaves reach my eyes, leading me to the construction of an image of the tree. In the same way, in speaking to you I am convinced that I press my opinions upon you, but you instruct me that in reality I produce vibrations in the air, which cause you to perform some action, e.g. to produce other vibrations. In both cases the first view is the natural one, the second is a theoretical construction. It is too often forgotten that the truth of such constructions depends upon the present state of

science and that the words "in reality" ought to be translated into "according to the contemporary view of scientists". Therefore I prefer to adhere to the idea that, when describing intuitionistic mathematics, I convey thoughts to my hearers; these words ought to be taken not in the sense of some philosophical system, but in the sense of every-day life.

SIGN. Then intuitionism, as a form of interaction between men, is a social phenomenon and its study belongs to the history of civilization.

INT. Its study, not its practice. Here I agree with Mr. Prag: primum vivere, deinde philosophari, and if we like we can leave the latter to others. Let those who come after me wonder why I built up these mental constructions and how they can be interpreted in some philosophy; I am content to build them in the conviction that in some way they will contribute to the clarification of human thought.

PRAG. It is a common fault of philosphers to speak about things they know but imperfectly and we are near to being caught in that trap. Is Mr. Int willing to give us some samples of intuitionistic reasoning, in order that we may better be able to judge the quality of the stuff?

INT. Certainly, and even I am convinced that a few lessons will give you a better insight into it than lengthy discussions. May I beg those gentlemen who are interested in my explanations, to follow me to my classroom?

VII

LOGIC

7.1. The propositional calculus

The word "logic" has many different meanings. I shall not try to give a definition of intuitionistic logic, any more than I have begun this course by a definition of mathematics. Yet a preliminary remark will be useful. Our logic has only to do with mathematical propositions; the question whether it admits any applications outside mathematics does not concern us here. The letters p, q, r occur in this chapter as variables for mathematical propositions; German letters $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$ will be used as abbreviations for mathematical propositions. It is not my purpose to give a complete formal treatment of intuitionistic logic; a formal system which codifies all the logical inferences of intuitionistic mathematics known at present, is easily accessible in Kleene's book [S. C. Kleene 1952], where the reader will also find an account of the metamathematical investigations of this system. Among later investigations I mention papers by E. W. Beth [1956, 1959], R. Harrop [1956, 1960], K. Schröter [1956, 1957], G. Kreisel and H. Putnam [1957], J. Porte [1958], H. Rasiowa and R. Sikorski [1959], R. Sikorski [1959], Th. Skolem [1958], S. C. Kleene [1962] and N. N. Vorob'ev [1958]. Kreisel [1958, 1962 A] formulated the completeness problem for intuitionistic logic and obtained important results on this subject. He also proposed a formal theory of constructions [1962] which is intended to serve as a basis for intuitionistic mathematics. Here I shall only call your attention to some formulas which express interesting methods of reasoning and show why these methods are intuitively clear within the realm of intuitionistic mathematics.

It will be necessary to fix, as firmly as possible, the meaning of the logical connectives; I do this by giving necessary and sufficient conditions under which a complex expression can be asserted.

7.1.1. *Interpretation of the signs*

The *conjunction* $\wedge$ gives no difficulty. $\mathfrak{p} \wedge \mathfrak{q}$ can be asserted if and only if both $\mathfrak{p}$ and $\mathfrak{q}$ can be asserted.

I have already spoken of the *disjunction* $\vee$ (2.2.5, at the end). $\mathfrak{p} \vee \mathfrak{q}$ can be asserted if and only if at least one of the propositions $\mathfrak{p}$ and $\mathfrak{q}$ can be asserted.

The *negation* $\neg$ is the strong mathematical negation which we have already discussed (2.2.2). In order to give a more explicit clarification, we remember that a mathematical proposition $\mathfrak{p}$ always demands a mathematical construction with certain given properties; it can be asserted as soon as such a construction has been carried out. We say in this case that the construction *proves* the proposition $\mathfrak{p}$ and call it a *proof* of $\mathfrak{p}$. We also, for the sake of brevity, denote by $\mathfrak{p}$ any construction which is intended by the proposition $\mathfrak{p}$. Then $\neg\mathfrak{p}$ can be asserted if and only if we possess a construction which from the supposition that a construction $\mathfrak{p}$ were carried out, leads to a contradiction.

SIGN. Is it not necessary to clarify the notion of a contradiction?

INT. I think that contradiction must be taken as a primitive notion. It seems very difficult to reduce it to simpler notions, and it is always easy to recognize a contradiction as such. In practically all cases it can be brought into the form $1 = 2$. As a simple example, let us consider the proposition $\mathfrak{p} \equiv (\sqrt{2}$ is rational). It demands the construction of integers a, b, such that $a^2 = 2b^2$. By a well-known argument we may suppose that a and b are relatively prime. On the other hand, a is even, so 4 divides a^2, hence 4 divides $2b^2$, and b is even; thus a and b have the common divisor 2. This contradicts the fact that a and b are relatively prime. The contradiction can be given the form: The GCD of a and b is at the same time 1 and 2.

Some mathematicians, and notably Griss, have raised objections against the use of contradiction in mathematical reasoning. I shall treat these objections in the next chapter; here I take the point of view that the notion of a contradiction is sufficiently clear and that the negation which is based on it can be used in mathematics.

The *implication* $\mathfrak{p} \to \mathfrak{q}$ can be asserted, if and only if we possess a construction $\mathfrak{r}$, which, joined to any construction proving $\mathfrak{p}$

(supposing that the latter be effected), would automatically effect a construction proving $\mathfrak{q}$. In other words, a proof of $\mathfrak{p}$, together with $\mathfrak{r}$, would form a proof of $\mathfrak{q}$.

Almost every proof in this book consists of such a construction as $\mathfrak{r}$ above. One of the first instances, and a very clear one, is the proof of 2.2.3, Th. 2.

A logical formula with proposition variables, say $\mathfrak{A}(p, q, \ldots)$, can be asserted, if and only if $\mathfrak{A}(\mathfrak{p}, \mathfrak{q}, \ldots)$ can be asserted for arbitrary propositions $\mathfrak{p}, \mathfrak{q}, \ldots$; that is, if we possess a method of construction which by specialization yields the construction demanded by $\mathfrak{A}(\mathfrak{p}, \mathfrak{q}, \ldots)$. For example consider

$$\mathfrak{A}(p, q) \equiv (p \wedge \cdot\ p \to q \cdot \to q).$$

$\mathfrak{A}(\mathfrak{p}, \mathfrak{q})$ demands a construction E, which from a proof C of $\mathfrak{p}$ and a proof D of $\mathfrak{p} \to \mathfrak{q}$ yields a proof of $\mathfrak{q}$. By the definition of implication, E consists simply in the juxtaposition of C and D. Thus $\mathfrak{A}(p, q)$ can be asserted.

In 2.2.2 I gave a criterion for mathematical propositions, namely that every mathematical proposition has the form "I have effected a construction with the following properties: .....". This form is preserved by the four logical connectives. It is necessary to understand the word "construction" in the wider sense, so that it can also denote a general method of construction, as was meant in the last paragraph but one. If I do this – and I shall do it –, every logical formula expresses a mathematical proposition.

7.1.2. *Theorems of the propositional calculus*

In the formulas I use points and brackets in the usual way, assuming the convention that $\to$ binds less strongly that $\wedge$ and $\vee$. Asserted formulas are marked with $\vdash$.

Though the main differences between classical and intuitionistic logic are in the properties of the negation, they do not coincide completely in their negationless formulas. $p \to q \cdot \vee \cdot q \to p$ is a valid formula in classical logic, but it cannot be asserted in intuitionistic logic, as is clear from the definitions.

In the theory of negation the principle of the excluded middle fails. $p \vee \neg p$ demands a general method to solve every problem, or more explicitly, a general method which for any proposition $\mathfrak{p}$

yields by specialization either a proof of $\mathfrak{p}$ or a proof of $\neg\mathfrak{p}$. As we do not possess such a method of construction, we have no right to assert the principle.

Another form of the principle is $\neg\neg p \to p$. We have met many examples of propositions for which this fails: the first was "ϱ is rational" in 2.2.2. However,

(1) $\qquad \vdash p \to \neg\neg p.$

It is clear that from $\mathfrak{p}$ it follows that it is impossible that $\mathfrak{p}$ is impossible. I leave it to you to describe completely the method of construction which is demanded by (1), according to the definitions of $\to$ and $\neg$.

Another important formula is

(2) $\qquad \vdash p \to q \cdot \to \cdot \neg q \to \neg p.$

Of course, the inverse formula, $\neg q \to \neg p \cdot \to \cdot p \to q$, is not assertable. (Take $\mathfrak{q} \equiv a \neq b$, $\mathfrak{p} \equiv a \neq b$, where a and b are real numbers.)

Applying (2) twice, we find

(3) $\qquad \vdash p \to q \cdot \to \cdot \neg\neg p \to \neg\neg q.$

By substitution in (1) we find

(4) $\qquad \vdash \neg p \to \neg\neg\neg p.$

If we substitute $\neg\neg p$ for q in (2), we find, using (1),

(5) $\qquad \vdash \neg\neg\neg p \to \neg p.$

(4) and (5) show that we need never consider more than two consecutive negations.

From $\vdash p \to p \vee q$ follows, by (2), $\vdash \neg(p \vee q) \to \neg p$; in the same way we have $\vdash \neg(p \vee q) \to \neg q$, so

(6) $\qquad \vdash \neg(p \vee q) \to \neg p \wedge \neg q.$

The inverse formula is easily seen to be also true:

(7) $\qquad \vdash \neg p \wedge \neg q \to \neg(p \vee q).$

(6) and (7) form one of de Morgan's equivalences. The other one is only half true:

(8) $\qquad \vdash \neg p \vee \neg q \to \neg(p \wedge q).$

$\neg (p \wedge q) \rightarrow \neg p \vee \neg q$ cannot be asserted, as the following example shows. Let $\mathfrak{p}$ be $a \neq 0$ and $\mathfrak{q}$ be $b \neq 0$, where a and b are real numbers; then $\neg \mathfrak{p}$ is $a = 0$ and $\neg \mathfrak{q}$ is $b = 0$. I proved in 2.2.5 that $ab \neq 0$ is equivalent to $\mathfrak{p} \wedge \mathfrak{q}$, so $ab = 0$ is equivalent to $\neg (\mathfrak{p} \wedge \mathfrak{q})$: but just before the cited place in 2.2.5 I gave an example of real numbers a and b for which $ab = 0$, but neither $a = 0$ nor $b = 0$ is known.

(9) $\quad \vdash \neg\neg (p \vee \neg p).$

For $\neg (\mathfrak{p} \vee \neg \mathfrak{p})$ would imply, by (6), $\neg \mathfrak{p} \wedge \neg\neg \mathfrak{p}$, which is a contradiction. (8) gives by means of (2) and (6)

$$\vdash \neg\neg (p \wedge q) \rightarrow \neg (\neg p \vee \neg q) \rightarrow \neg\neg p \wedge \neg\neg q.$$

(10) $\quad \vdash \neg\neg (p \wedge q) \rightarrow \neg\neg p \wedge \neg\neg q.$

The inverse formula is also true:

(11) $\quad \vdash \neg\neg p \wedge \neg\neg q \rightarrow \neg\neg (p \wedge q).$

For it is clear from the above interpretation of the logical connectives that $\vdash \neg (p \wedge q) \wedge p \rightarrow \neg q$; then also $\vdash \neg (p \wedge q) \wedge \neg\neg q \rightarrow \neg p$. So, if $\neg\neg \mathfrak{p} \wedge \neg\neg \mathfrak{q}$ is given, the hypothesis $\neg (\mathfrak{p} \wedge \mathfrak{q})$ would lead to $\neg \mathfrak{p}$, which is contradictory with the given $\neg\neg \mathfrak{p}$.
It is easy to see that

(12) $\quad \vdash \neg\neg p \vee \neg\neg q \rightarrow \neg\neg (p \vee q),$

but the inverse implication does not hold because of the strong interpretation of $\vee$.

7.1.3. *A formal system*

The intuitionistic propositional calculus has been developed [A. Heyting 1930] as a formal system with $\wedge$, $\vee$, $\rightarrow$, $\neg$ as undefined constants, and on the basis of the following axioms

I. $\quad \vdash p \rightarrow (p \wedge p).$

II. $\quad \vdash (p \wedge q) \rightarrow (q \wedge p).$

III. $\quad \vdash (p \rightarrow q) \rightarrow ((p \wedge r) \rightarrow (q \wedge r)).$

IV. $\quad \vdash ((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r).$

V. $\quad \vdash q \rightarrow (p \rightarrow q).$

VI. $\quad \vdash (p \wedge (p \rightarrow q)) \rightarrow q.$

VII. $\vdash p \to (p \vee q)$.

VIII. $\vdash (p \vee q) \to (q \vee p)$.

IX. $\vdash ((p \to r) \wedge (q \to r)) \to ((p \vee q) \to r)$.

X. $\vdash \neg p \to (p \to q)$.

XI. $\vdash ((p \to q) \wedge (p \to \neg q)) \to \neg p$.

The rules of deduction are the usual ones from the classical propositional calculus.

Axiom X may not seem intuitively clear. As a matter of fact, it adds to the precision of the definition of implication. You remember that $\mathfrak{p} \to \mathfrak{q}$ can be asserted if and only if we possess a construction which, joined to the construction $\mathfrak{p}$, would prove $\mathfrak{q}$. Now suppose that $\vdash \neg \mathfrak{p}$, that is, we have deduced a contradiction from the supposition that $\mathfrak{p}$ were carried out. Then, in a sense, this can be considered as a construction, which, joined to a proof of $\mathfrak{p}$ (which cannot exist) leads to a proof of $\mathfrak{q}$. I shall interpret the implication in this wider sense.

A system of intuitionistic logic in which $\to$ is interpreted in the narrower sense and in which, accordingly, X is rejected as an axiom, has been developed by Johansson in his "minimal calculus" [I. Johansson 1936].

It must be remembered that no formal system can be proved to represent adequately an intuitionistic theory. There always remains a residue of ambiguity in the interpretation of the signs, and it can never be proved with mathematical rigour that the system of axioms really embraces every valid method of proof.

7.2. The first order predicate calculus

7.2.1. *Interpretation of the quantifiers*

Let $\mathfrak{p}(x)$ be a predicate of one variable x, this variable ranging over a given mathematical species Q. Then $\vdash (\forall x)\mathfrak{p}(x)$ means that $\mathfrak{p}(x)$ is true for every x in Q; in other words, we possess a general method of construction which, if any element a of Q is chosen, yields by specialization the construction $\mathfrak{p}(a)$. In the case that Q is a spread-species, we must be able to effect the construction $\mathfrak{p}(x)$ for every ips x in Q; in the proof of the fan-theorem we saw that this is a very strong interpretation of the generalizing quantifier. The existential quantifier will also be interpreted in a

strong way. $(\exists x)\mathfrak{p}(x)$ will be true if and only if an element a of Q for which $\mathfrak{p}(a)$ is true has actually been constructed.

The introduction of predicates with more than one argument presents no difficulty. A formula of the first order predicate calculus, which contains propositional and predicate variables, can be asserted if it is true for every substitution of propositions and predicates for these variables. A simple formalization of the intuitionistic predicate calculus is obtained by adjoining to the intuitionistic propositional calculus the symbols, axioms and rules of the usual predicate calculus as stated by Hilbert and Ackermann ([D. Hilbert und W. Ackermann 1949, p. 59]; see also [A. Heyting 1946]). I shall not develop this formal system, but instead I shall prove some formulas by intuitive methods.

7.2.2. *Theorems of the predicate calculus*

The following theorems are clear.

(1) $\quad \vdash (\forall x)p(x) \to \neg\, (\exists x) \neg\, p(x).$

(2) $\quad \vdash (\exists x)p(x) \to \neg\, (\forall x) \neg\, p(x).$

The inverse implications do not hold. Counterexamples:

(i) Let x range over the real numbers and let $\mathfrak{p}(x)$ be "x is rational or x is irrational".

(ii) Let x range over the rational numbers and let $\mathfrak{p}(x)$ be "x is equal to the real number ϱ, defined in 2.2.2.".

(3) $\quad \vdash (\forall x) \neg\, p(x) \to \neg\, (\exists x)p(x).$

(4) $\quad \vdash \neg\, (\exists x)p(x) \to (\forall x) \neg\, p(x).$

(5) $\quad \vdash (\exists x) \neg\, p(x) \to \neg\, (\forall x)p(x).$

The inverse implication of (5) does not hold.
Counterexample:
Let x range over the rational numbers and let $\mathfrak{p}(x)$ be "x is not equal to the real number ϱ, defined in 2.2.2.".
By substitution in (3) we obtain

(6) $\quad \vdash (\forall x) \neg\neg\, p(x) \to \neg\, (\exists x) \neg\, p(x).$

Substitution in (4) gives

(7) $\quad \vdash \neg\, (\exists x) \neg\, p(x) \to (\forall x) \neg\neg\, p(x).$

Applying the formulas of the propositional calculus 7.1.2. (3) and (5) to (6) of this section we obtain

(8) $\quad \vdash \neg\neg\,(\forall x)\neg\neg\, p(x) \to \neg\,(\exists x)\neg\, p(x).$

This can be weakened to

(9) $\quad \vdash \neg\neg\,(\forall x)p(x) \to \neg\,(\exists x)\neg\, p(x).$

(9) and (7) yield the important result

(10) $\quad \vdash \neg\neg\,(\forall x)p(x) \to (\forall x)\neg\neg\, p(x).$

It is one of the most striking features of intuitionistic logic that the inverse implication does not hold, especially because the formula of the propositional calculus which results if we restrict x to a finite set, is true. In fact, if x takes only two values we obtain 7.1.2, (11). Brouwer [L. E. J. Brouwer 1924, p. 256; A. Heyting 1930A, p. 65; S. C. Kleene 1952, p. 491] gave the following counter-example:

A spread M is defined by the spreadlaw Λ_M and the complementary law Γ_M.

Λ_M. The first component of an admissible sequence can be 0 or 1. If $a_1, \ldots, a_n$ is an admissible sequence and $a_n = 0$, then a_{n+1} may be 0 or 1; if $a_n = 1$, then $a_{n+1} = 1$.

Γ_M assigns a_n to the admissible sequence $a_1, \ldots, a_n$.

In plain words, M consists of all ipss in which only 0 and 1 can be components, while 1 can only be followed by 1.

The following law Ξ assigns numbers to certain elements of M.

Ξ: The sequence, all of whose components are 0, has number 1. The sequence, all of whose components are 1, has number 2. The sequence which consists of n components 0 followed by components 1 has number $n+2$.

Let x range over the elements of M and let $\mathfrak{p}(x)$ be "Ξ assigns a number to x". Then $(\forall x)\neg\neg\, p(x)$ is true. I prove the equivalent proposition, $\neg\,(\exists x)\neg\, \mathfrak{p}(x)$, by deducing a contradiction from the supposition that Ξ assigns no number to a certain element a of M. Under this supposition, the first component of a cannot be 1, for then $\Xi(a)$ would be 2, so the first component is 0. The second component cannot be 1, for then $\Xi(a)$ would be 3, so the second component is 0. Continuing in this way we see that every component of a is 0; it follows that $\Xi(a) = 1$, a result which contradicts the supposition.

Yet, $\neg\neg(\forall x)\mathfrak{p}(x)$ does not hold; we can even prove that $\neg(\forall x)\mathfrak{p}(x)$. Supposing that $(\forall x)\mathfrak{p}(x)$, the fan-theorem allows us to infer that $\Xi(x)$ must be known after a finite number N of components of x are given, but this clearly contradicts the definition of Ξ.

CLASS. This is the first instance of a classical theorem which is intuitionistically not only unprovable, but even false.

INT. Of course, this divergence is brought about by the introduction of ipss; especially, the generalizing quantifier has quite a different meaning if applied to ipss.

It has been conjectured [S. Kuroda 1951, p. 46] that the formula $(\forall x)\neg\neg p(x) \to \neg\neg(\forall x)p(x)$ is always true if x ranges over a denumerable infinite species. In all the counterexamples that have been given, x ranges over a species that is not denumerably infinite (this holds for M above, though M is denumerable from the classical point of view!), but no way of proving the conjecture presents itself at present.

From (3) and (4) we infer, applying 7.1.2,(2):

(11) $\quad \vdash \neg\neg(\exists x)p(x) \to \neg(\forall x)\neg p(x).$

(12) $\quad \vdash \neg(\forall x)\neg p(x) \to \neg\neg(\exists x)p(x).$

By substitution in (5):

(13) $\quad \vdash (\exists x)\neg\neg p(x) \to \neg(\forall x)\neg p(x).$

From (13) and (12):

(14) $\quad \vdash (\exists x)\neg\neg p(x) \to \neg\neg(\exists x)p(x).$

Here also the inverse implication is not valid, but the fact is less surprising than for (10), because the corresponding formula of the propositional calculus, which would be $\neg\neg(p \vee q) \to \neg\neg p \vee \neg\neg q$, is also not provable.

7.3. Applications

In some cases a negative theory can be developed parallel to a positive theory. In the latter every notion is defined by a positive definition; in the former some notions are introduced by negative definitions, involving double negations, but in such a way that

definitions of corresponding notions in the two theories are classically equivalent.

7.3.1. *Order relations in the continuum*

Definition 1. If S is a mathematical species and if $<$ is a predicate, defined on a subspecies of $S \times S$, which satisfies rules (1)–(4) below, then $<$ is a *partial order relation* in S and S is *partially ordered* by $<$. ($a>b$ means the same as $b<a$.)

(1) $a<b \to \neg\,(a>b) \wedge \neg\,(a=b)$.

(2) $a=b \wedge b<c \to a<c$.

(3) $a<b \wedge b=c \to a<c$.

(4) $a<b \wedge b<c \to a<c$.

Definition 2. A partial order relation in a species S is an *order relation* in S and S is *ordered* by it, if it satisfies rule (5a).

(5a) $a=b \vee a<b \vee a>b$.

The relation $<$ between natural numbers or rational numbers is an order relation. The relation $<$ between real numbers, as defined in 2.2.6. in connection with 3.3, is not an order relation; this follows immediately from 3.4.5, Th. 2.

Instead of (5a), it satisfies (5b) and (6), which are both consequences of (1)–(4), (5a).

(5b) $\neg\,(a>b) \wedge \neg\,(a<b) \to a=b$ (Compare 2.2.6, Th. 3).

(6) $a<b \to (\forall c)(a<c \vee c<b)$ (Compare 2.2.6, Th. 4).

Definition 3. A partial order relation in a species S is a *pseudo-order relation* and S is *pseudo-ordered* by it, if it satisfies rules (5b) and (6). (Note that (2) and (3) follow from (1) and (6).)

In the following $<$ always denotes a pseudo-order relation in a species S.

Definition 4. $a \lesssim b$ means $\neg\,(a>b) \wedge \neg\,(a=b)$.

Theorem 1. $\neg\,(a \lesssim b)$ is equivalent to $\neg\,(a<b)$.

Proof. By (1), $a<b \to a \lesssim b$, so $\neg\,(a \lesssim b) \to \neg\,(a<b)$. (7.1.2, (2)). To prove the converse, I remark that $a \lesssim b \wedge \neg\,(a<b)$ means the same as $\neg\,(a>b) \wedge \neg\,(a<b) \wedge \neg\,(a=b)$, which contradicts (5b). So $a \lesssim b \wedge \neg\,(a<b)$ is impossible; consequently $\neg\,(a<b) \to \neg\,(a \lesssim b$.

Theorem 2. $a \lesssim b$ is equivalent to $\neg\neg\,(a<b)$.

Proof. By def. 4 and 7.1.2, (6), (7), $a \prec b$ is equivalent to $\neg\,(a>b \vee a=b)$, so
$\neg\neg\,(a \prec b) \to a \prec b$ (7.1.2, (5)).
$\neg\,(a \prec b) \to \neg\,(a<b)$ by Th. 1, so
$\neg\neg\,(a<b) \to \neg\neg\,(a \prec b)$ (7.1.2, (2)) $\to a \prec b$.
$\neg\,(a<b) \to \neg\,(a \prec b)$ by Th. 1, so
$a \prec b \to \neg\neg\,(a<b)$ (7.1.2, (1), (2)).

Theorem 3. The relation $\prec$ satisfies (1)–(4), (5b).

Proof. (1) follows from def. 4 and Th. 1.
(2). From (2) for $<$, by 7.1.2, (3) and Th. 2, as follows:

$$a=b \wedge \neg\neg\,(b<c) \to \neg\neg\,(a<c) \to a \prec c.$$

(3). Analogously.
(4). From (4) for $<$ by 7.1.2, (11), (3) and Th. 2.
(5b) follows from Th. 1 and (5b) for $<$.

In the case of the continuum $\prec$ does not satisfy (6) (see 8.1.1). Instead, it satisfies (5c):

(5c) $\neg\,(a \prec b) \wedge \neg\,(a=b) \to a \succ b$.

Theorem 4. If $<$ is a pseudo-order relation in a species S, the relation $\prec$ which corresponds to it by def. 4, satisfies (5c).

Proof. Immediately from Th. 1 and def. 4.

Definition 5. A partial order relation in a species S which satisfies (1)–(4), (5b), (5c), is a *virtual order relation* in S and S is *virtually ordered* by it [L. E. J. Brouwer 1925A, p. 453].

Theorems 3 and 4 can now be expressed as follows. If $<$ is a pseudo-order relation in a species S, then $\prec$, as defined by def. 4, is a virtual order relation in S.

Remark on notation. My notation is different from Brouwer's. Here follows a small dictionary.

This book:	$<$	$\not<$	$\prec$
Brouwer :	$<_0$	$\geqslant$	$<$

The reason for changing the notation was, that in analysis the pseudo-order relation between real numbers is the most important; therefore, it should be denoted by the simplest sign. Also in Brouwer's notation there is danger of confusion between $a \geqslant b$ and $(a>b \vee a=b)$. I use $a \geqq b$ in the latter sense.

According to def. 5, the rules for a virtual order relation are:

(1) $a \prec b \to \neg (a \succ b) \wedge \neg (a = b)$.

(2) $a = b \wedge b \prec c \to a \prec c$.

(3) $a \prec b \wedge b = c \to a \prec c$.

(4) $a \prec b \wedge b \prec c \to a \prec c$.

(5b) $\neg (a \prec b) \wedge \neg (a \succ b) \to a = b$.

(5c) $\neg (a \prec b) \wedge \neg (a = b) \to a \succ b$.

Theorem 5. If $\prec$ is a virtual order relation in a species S, then $\neg\neg (a \prec b) \to a \prec b$ and $\neg\neg (a = b) \to a = b$.

Proof. By (1) and (5c), $a \prec b$ is equivalent to $\neg (a \succ b) \wedge \neg (a = b)$, which, by 7.1.2, (6), (7), is equivalent to $\neg (a \succ b \vee a = b)$. Thus, by 7.1.2, (5), $\neg\neg (a \prec b) \to a \prec b$.

From (1) and 7.1.2, (2), (1) we infer that $a = b \to \neg (a \prec b)$. Similarly, $a = b \to \neg (a \succ b)$. Thus, using (5b) and 7.1.2, (6), (7), we see that $a = b$ is equivalent to $\neg (a \prec b \vee a \succ b)$. This proves that $\neg\neg (a = b) \to a = b$.

Definition 6. Let $^\circ\!\prec$ be a partial order relation in the species S, and let $\sum$ be the species of formulas $a = b$ or $a \mathrel{\overset{\circ}{<}} b$, which are valid for elements a, b of S. $\overset{\circ}{<}$ is called *unextensible* if it has the following properties: Whenever $x, y \in S$ and $\neg (x \overset{\circ}{<} y)$ cannot be deduced from $\sum$ by applications of (1)–(4), then $x \overset{\circ}{<} y \in \sum$; similarly, whenever $x, y \in S$ and $\neg (x = y)$ cannot be deduced from $\sum$ by applications of (1)–(4), then $x = y \in \sum$.

Theorem 6. Every virtual order relation is unextensible [L. E. J. Brouwer 1927].

Proof. If $x \succ y \in \sum$, then $\neg (x \prec y)$ can be deduced from $\sum$ by (1), so if $\neg (x \prec y)$ cannot be deduced from $\sum$ by (1)–(4), then $\neg (x \succ y \in \sum)$, and similarly, $\neg (x = y \in \sum)$.
Thus, by (5c), $x \prec y \in \sum$.

The proof for $=$ is analogous.

Theorem 7. Every unextensible partial order relation is a virtual order relation.
For the proof, see Brouwer [L. E. J. Brouwer 1927].

7.3.2. *The negative theory of convergence*

2.4, Def. 1 can be formulated as follows:
The sequence $\{a_n\}$ of real numbers is positively convergent to the limit a if

(1) $\quad (\forall k)\,(\exists n)\,(\forall p)\,(|a-a_{n+p}|<2^{-k})$.

As the corresponding negative definition I choose:

Definition 1. The sequence $\{a_n\}$ is *negatively convergent* to the limit a ($^{-}\lim a_n=a$) if

(2) $\quad (\forall k)\,\neg\neg\,(\exists n)\,(\forall p)\,(|a-a_{n+p}|<2^{-k})$ [J. G. Dijkman 1952].

Cauchy's general criterion for convergence is

(3) $\quad (\forall k)\,(\exists n)\,(\forall p)\,(|a_{n+p}-a_n|<2^{-k})$.

Definition 2. The sequence $\{a_n\}$ is *non-oscillating*, if

(4) $\quad (\forall k)\,\neg\neg\,(\exists n)\,(\forall p)\,(|a_{n+p}-a_n|<2^{-k})$.

It is well-known that if a sequence $\{a_n\}$ satisfies (3), then a real number a can be found so that (1) is true. The corresponding relation does not hold between (4) and (2), as example (ii) shows.

Examples. (i) $a_n=1$ if the n-th digit after the decimal point in the decimal expansion of π is the 9 of the first sequence 0123456789 in this expansion (let us call this value of n, if it exists, the sequence-number); otherwise $a_n=0$. In order to show that this sequence satisfies (2) with $a=0$, I first remark that (2) is equivalent to

(5) $\quad \neg\,(\exists k)\,(\forall n)\,\neg\,(\forall_p)\,(|a-a_{n+p}|<2^{-k})$.

Suppose we had found k_0 such that

(α) $\quad (\forall n)\neg\,(\forall p)\,(|a_{n+p}|<2^{-k_0})$.

Suppose further that a sequence occurs in π and that s is the sequence-number; then

(β) $\quad (\forall p)\,(a_{s+p}=0)$, so $(\forall p)\,(|a_{s+p}|<2^{-k_0})$.

So (α) cannot be true. We have now proved:
If (α) is true, then no sequence 0123456789 can occur in π; but then $(\forall n)\,(a_n=0)$, and (α) is false. Thus, (α) has led to a contra-

diction, and (5) is proved. Yet it is clear that we cannot assert that the sequence $\{a_n\}$ is positively convergent.

(ii) $a_n = 0$ if no sequence 0123456789 occurs in the first n digits after the decimal point in π; $a_n = 1$ if such a sequence does occur in the first n digits. This sequence $\{a_n\}$ is non-oscillating; the proof is analogous to the preceding one. Yet we cannot assert that it is negatively convergent, for it is unknown whether a limit, if it exists, will be 0 or 1.

Theorem. Every negatively convergent sequence is non-oscillating.

Proof. Let us abbreviate

$(\exists n)\,(\forall p)\,(|a_{n+p} - a_n| < 2^{-k})$ by $\mathfrak{A}(k)$, and

$(\exists n)\,(\forall p)\,(|a_{n+p} - a\,| < 2^{-k})$ by $\mathfrak{B}(k, a)$. Then $\mathfrak{B}(k+1, a) \to \mathfrak{A}(k)$.

$\neg\neg\,\mathfrak{B}(k+1, a) \to \neg\neg\,\mathfrak{A}(k)$.

$(\forall k)\,\neg\neg\,\mathfrak{B}(k, a) \to (\forall k)\,\neg\neg\,\mathfrak{A}(k)$

$(\exists a)\,(\forall k)\,\neg\neg\,\mathfrak{B}(k, a) \to (\forall k)\,\neg\neg\,\mathfrak{A}(k)$.

This proves the theorem.

Theorem 2. If $^{-}\lim a_n = a$ and $^{-}\lim b_n = b$, then $^{-}\lim (a_n + b_n) = a + b$.

Proof. Set $a_n + b_n = c_n$, $a + b = c$.

Let us abbreviate

$(\exists n)\,(\forall p)\,(|a_{n+p} - a| < 2^{-k})$ by $\mathfrak{A}(k)$,

$(\exists n)\,(\forall p)\,(|b_{n+p} - b| < 2^{-k})$ by $\mathfrak{B}(k)$,

$(\exists n)\,(\forall p)\,(|c_{n+p} - c| < 2^{-k+1})$ by $\mathfrak{C}(k)$.

Then $\mathfrak{A}(k) \wedge \mathfrak{B}(k) \to \mathfrak{C}(k)$, thus $\neg\neg\,\mathfrak{A}(k) \wedge \neg\neg\,\mathfrak{B}(k) \to \neg\neg\,\mathfrak{C}(k)$ (Application of 7.1.2, (11) and (3)).

$(\forall k)\,\neg\neg\,\mathfrak{A}(k) \wedge (\forall k)\,\neg\neg\,\mathfrak{B}(k) \to (\forall k)\,\neg\neg\,\mathfrak{C}(k)$.

This proves the theorem.

Theorem 3. If $^{-}\lim a_n = a$ and $^{-}\lim a_n = b$, then $a = b$.

Proof. From $^{-}\lim a_n = b$ it follows that $^{-}\lim -a_n = -b$. Thus by Th. 2, $^{-}\lim c_n = a - b$, where every $c_n = 0$.

$(\forall k) \neg\neg (\exists n) (\forall p) (|a-b| < 2^{-k})$.
$(\forall k) \neg\neg (|a-b| < 2^{-k})$.
$(\forall k) (|a-b| \not> 2^{-k})$.
$a = b$.

The following theorem has no parallel in the positive theory.

Theorem 4. Every bounded monotone sequence of real numbers is non-oscillating.

For the sequence $\{a_n\}$, we have by hypothesis,

(1) $(\forall n) (a_n < M)$,

(2) $(\forall n) (a_{n+1} \not< a_n)$.

It must be proved that

(X) $(\forall k) \neg\neg (\exists n) (\forall p) (a_{n+p} - a_n < 2^{-k})$,

which is equivalent to

(Y) $\neg (\exists k) (\forall n) \neg (\forall p) (a_{n+p} - a_n < 2^{-k})$.

Let us suppose that we have found a number k_1 such that

(5) $(\forall n) \neg (\forall p) (a_{n+p} - a_n < 2^{-k_1})$.

It suffices to deduce a contradiction from (1), (2) and (5). Now from $\neg (\exists p) (a_{n+p} - a_n > 2^{-k_1-1})$, it follows that $(\forall p) (a_{n+p} - a_n < 2^{-k_1})$, so from (5) it follows that

(6) $(\forall n) \neg\neg (\exists p) (a_{n+p} - a_n > 2^{-k_1-1})$.

The theorem will be proved if we deduce a contradiction from (1), (2) and (6).

As a special case of (6) we have

(7) $\neg\neg (\exists p) (a_p - a_1 > 2^{-k_1-1})$.

Let us suppose that we have found a number p_1 such that $a_{p_1} - a_1 > 2^{-k_1-1}$; then, by (6), $\neg\neg (\exists r) (a_r - a_{p_1} > 2^{-k_1-1})$, and further $\neg\neg (\exists r) (a_r - a_1 > 2.2^{-k_1-1})$.

We have now proved from (1), (2) and (6),

(8) $(\exists p) (a_p - a_1 > 2^{-k_1-1}) \to \neg\neg (\exists r) (a_r - a_1 > 2.2^{-k_1-1})$.

Using the propositional calculus (7.2.1, (3), (4), (2)), we infer from

(7) and (8) that

(9) $\neg\neg(\exists r)(a_r - a_1 > 2.2^{-k_1-1})$.

By repeating $h-1$ times this reasoning, we find

(10) $\neg\neg(\exists s)(a_s - a_1 > h.2^{-k_1-1})$.

Now take $h = \left[\frac{M-a_1}{2^{-k_1-1}}\right]+1$; we find

(11) $\neg\neg(\exists s)(a_s > M)$.

This contradicts (1).

Remark. It is easily seen that instead of (1), the hypothesis $\neg\neg(\exists M)(\forall n)(a_n < M)$ is sufficient.

7.3.3. *Negative interpretation of classical analysis*

It was first proved by Gödel [K. Gödel 1932] that the classical propositional calculus and classical arithmetic can be developed as parts of the corresponding intuitionistic systems. In order to establish such theorems it is necessary first to formalize the relevant part of intuitionistic mathematics. The remarks which I made concerning the intuitionistic calculus of propositions apply to every formal system which is constructed with the intention of representing an intuitionistic mathematical theory: it can never be proved rigorously that such a system is adequate. However, formal systems have been constructed for the propositional calculus, predicate calculus and elementary arithmetic, such that every provable formula of one of these systems, if interpreted in the right way, expresses a theorem of intuitionistic mathematics. These systems are described in detail by Kleene in his book [S. C. Kleene 1952, p. 492]; he gives them in such a way that, by adjoining to each of them the axiom $\neg\neg p \to p$, we obtain the corresponding classical systems.

Kleene proves various extensions of Gödel's theorem, of which I mention the theorems 1, 2 and 3 below.

Definition 1. $p+q$ means $\neg(\neg p \wedge \neg q)$.
$p \supset q$ means $\neg(p \wedge \neg q)$.
$(\forall x)p(x)$ means $\neg(\forall x)\neg p(x)$.

By 7.3.1. (3) and (4), the latter is equivalent to $\neg\neg(\exists x)p(x)$.

Theorem 1. For the propositional calculus with $\wedge$ as conjunction, $+$ as disjunction, $\supset$ as implication and $\neg$ as negation, every classically provable formula is true intuitionistically.

Theorem 2. For the number-theoretic formal system (elementary arithmetic) with $\wedge$ as conjunction, $+$ as disjunction, $\supset$ or $\rightarrow$ as implication, $\neg$ as negation, and $(\forall\ )$ as existential quantifier, every classically provable formula is true intuitionistically.

Theorem 3. For the predicate calculus, with $\wedge$ as conjunction, $+$ as disjunction, $\supset$ or $\rightarrow$ as implication, $\neg$ as negation, and $(\forall\ )$ as existential quantifier, every classically provable formula becomes true intuitionistically after every prime formula $\mathfrak{p}$ which occurs in it, has been replaced by $\neg\neg\mathfrak{p}$.

Remark. The transformation of $\mathfrak{p}$ into $\neg\neg\mathfrak{p}$ is unnecessary in the propositional calculus, because $\neg\neg\mathfrak{p}\supset\mathfrak{p}$, that is $\neg(\neg\neg\mathfrak{p}\wedge\neg\mathfrak{p})$, is true, and in the number-theoretic system, because the prime formulas have the form $a=b$ or $a>b$, where a and b are natural numbers, so that $\neg\neg(a=b)\rightarrow a=b$ and $\neg\neg(a>b)\rightarrow a>b$ are true.

These theorems contain consistency proofs for the classical systems relatively to the corresponding intuitionistic systems.

I shall now apply these theorems to the arithmetic of real numbers. A number-generator was defined as a Cauchy sequence of rational numbers. For the application of Th. 2, in the definition of a Cauchy sequence, $(\forall x)$ must be substituted for $(\exists x)$, so that we obtain the definition of a non-oscillating sequence.

Definition 2. A *weak real number-generator* is a non-oscillating sequence of rational numbers.

The definition of coincidence for real number-generators was as follows: If $a=\{a_n\}$ and $b=\{b_n\}$, then $a=b$ means that

$$(\forall k)\,(\exists n)\,(\forall p)\,(|a_{n+p}-b_{n+p}|<2^{-k}).$$

Definition 3. The weak number-generators $a=\{a_n\}$ and $b=\{b_n\}$ *coincide in the weak sense* $(a\doteq b)$ if

$$(\forall k)\,(\forall n)\,(\forall p)\,(|a_{n+p}-b_{n+p}|<2^{-k}).$$

Definition 4. A *weak real number* is the species of the weak

number-generators which coincide in the weak sense with a given weak number-generator.

The definition of the order relation between real number-generators was as follows: $a > b$ means that

$$(\exists k)\,(\exists n)\,(\forall p)\,(a_{n+p} - b_{n+p} > 2^{-k}).$$

Definition 5. For weak number-generators $a = \{a_n\}$ and $b = \{b_n\}$, *a is weakly greater than b* ($a \succ b$) means that

$$(\forall k)\,(\forall n)\,(\forall p)\,(a_{n+p} - b_{n+p} > 2^{-k}).$$

This is equivalent to the double negation of $a > b$, for from 7.2.2, (11) and (12) we infer easily that $\neg\neg\,(\exists x)p(x)$ is equivalent to $\neg\neg\,(\exists x)\,\neg\neg\,p(x)$.

As a corollary of Th. 2 we have

Theorem 4. Let S be the system which is obtained from the number-theoretic formal system by adjoining to it free variables for real numbers and the relations $=$ and $>$ for real numbers. Let T be a theorem of classical arithmetic which can be formalized in S. Let T' be the theorem which is obtained from T by replacing $\vee$ by $+$, $(\exists x)$ by $(\forall x)$, the variables for real numbers by variables for weak real numbers, $=$ by $\doteq$ and $>$ by $\succ$, then T' is true intuitionistically.

In order to extend theorem 4 to analysis, it will be necessary to extend Th. 2 or Th. 3 to a predicate calculus of higher order or to some other calculus in which the main part of classical analysis can be formalized. Probably a consistency proof for analysis relatively to intuitionistic mathematics can be derived by this method.

VIII

CONTROVERSIAL SUBJECTS

8.1. Infinitely proceeding sequences, depending upon the solving of problems

8.1.1. *The method*

Since 1948 Brouwer has published a number of papers, many of them in Dutch [L. E. J. Brouwer 1948, p. 1246; 1948A; 1948B; 1949; 1949A; 1950; 1950A; 1951; 1952B; 1954B; 1954C], in which he gives counterexamples to classical theorems. All these examples are based on a principle which he indicated in his conference for the 10-th Int. Congress of Philosophy, Amsterdam 1948. I begin by illustrating it by means of the following definitions.

We shall say that a mathematical proposition $\mathfrak{p}$ has been *tested* if either $\neg\,\mathfrak{p}$ or $\neg\neg\,\mathfrak{p}$ has been proved. Let $\mathfrak{p}$ be a proposition which has not been tested (e.g. "a sequence 0123456789 occurs in the decimal expansion of π"). Now I define a real number-generator $a=\{a_n\}$ by an ips as follows. As long as $\mathfrak{p}$ has not been tested, I choose $a_n=2^{-n}$, but if $\mathfrak{p}$ is tested between the choice of a_m and that of a_{m+1}, then I choose $a_{m+q}=2^{-m}$ for every q.

SIGN. This does not sound like a mathematical definition. Can a sequence of rational numbers be considered as mathematically well-defined if its components depend upon material facts, such as the existence at a given moment of a proof for a certain proposition?

INT. I agree to this objection; and, indeed, I doubt whether it is advisable to adopt such definitions as mathematical. As I stressed before, no verbal definition can be perfectly unambiguous; we see now that the definition of an ips left us some free play. In such a case we may decide freely which interpretation we adopt.

FORM. As van Dantzig has remarked [D. van Dantzig 1949], Brouwer's definition and the reasoning which he based on it can be fully justified from a formal point of view.

INT. Van Dantzig's methods, as he indicates, also throw some light on the subject as it is seen by the intuitionists. I shall try to characterize briefly his point of view.

Let ω_i $(i=1, 2, \ldots)$ denote for every i a finite set of mathematical deductions. The sequence $\{\omega_i\}$ will be denoted by Ω.

Set $\sigma_n = \bigcup_{i=1}^{n} \omega_i$. Let $\mathfrak{p}$ be a mathematical proposition. The real number-generator a depends upon Ω; $a(\Omega) = \{a_n(\Omega)\}$. If σ_n contains no deduction of either $\neg\, \mathfrak{p}$ or $\neg\neg\, \mathfrak{p}$, then $a_n(\Omega) = 2^{-n}$. If σ_n contains a deduction of $\neg\, \mathfrak{p}$ or of $\neg\neg\, \mathfrak{p}$, and if m is the least number such that σ_m contains a deduction of $\neg\, \mathfrak{p}$ or of $\neg\neg\, \mathfrak{p}$, then $a_n(\Omega) = 2^{-m}$. In this form the definition of a sounds more mathematical; the question remains whether it is adequate for Brouwer's purpose.

Brouwer wished to show that $a \neq 0$ while $a \# 0$ has not been proved (This is the example which I promised in 2.2.3). For suppose $a = 0$; then neither $\neg\, \mathfrak{p}$ nor $\neg\neg\, \mathfrak{p}$ could ever be proved, so $\neg\neg\, \mathfrak{p}$ and $\neg\neg\neg\, \mathfrak{p}$ would both be true, a contradictory result. Hence $a \neq 0$. On the other hand, $a \# 0$ would mean that we could find a number q so that $a > 2^{-q}$; it follows that $\mathfrak{p}$ would be tested before the choice of a_q. We can only know this if $\mathfrak{p}$ has already been tested. Note that $a \# 0$ is not contradictory, for this would imply $a = 0$.

Evidently Brouwer supposes that he does not know beforehand which deductions will be made; if a law is passed throughout the world prohibiting the making of any mathematical deduction whatever, then the proof of $a \neq 0$ fails.

In van Dantzig's version the result is as follows. For a given n, the assertion, that $a = 0$ whatever $\omega_{n+1}, \omega_{n+2}, \ldots$ may be, is false. If $a \# 0$, then a number q is known such that a proof of either $\neg\, \mathfrak{p}$ or $\neg\neg\, \mathfrak{p}$ occurs in σ_q.

SIGN. That is to say, in van Dantzig's terminology we have no example of a real number a such that $a \neq 0$ while $a \# 0$ is unproved.

INT. That is so. Personally, I prefer van Dantzig's terminology, which avoids several equivocalities in Brouwer's words. Van Dantzig acutely analyses Brouwer's definition and introduces many refinements, for which I refer to his paper. In my opinion it is not very important whether we express the result in Brouwer's words or in those of van Dantzig, provided we understand what is meant by

it. Nor is it important whether we call it a mathematical result or not. In any case it shows that it would be foolish to seek a proof for the equivalence of the relations $\neq$ and $\#$ between real numbers.

CLASS. I have been convinced of that since 2.2.3.

INT. Brouwer's example analyses one of the subconscious reasons which made you feel this conviction.

Brouwer's example can also be used to demonstrate that the virtual order of the continuum is not a pseudo-order (see 7.3.1). Let me repeat the example. Let $\mathfrak{p}$ be a proposition which has not been tested. I define a real number-generator $a=\{a_n\}$ by an ips as follows. As long as $\mathfrak{p}$ has not been tested, I choose $a_n=2^{-n}$, but if $\mathfrak{p}$ is tested between the choice of a_m and that of a_{m+1}, then I choose $a_{m+q}=2^{-m}$ for every q. Now I define $c=\{c_n\}$ as follows. As long as $\mathfrak{p}$ has not been tested, I choose $c_n=2^{-n}$, but if $\mathfrak{p}$ is tested between the choice of a_m and that of a_{m+1}, *and m is even*, then I choose $c_{m+q}=2^{-m}$ for every q, but if m is odd, then I choose $c_r=2^{-r}$ even for $r>m$.

If $c\#0$, then $c=a$; hence, if $a>c$, then $c=0$. If $c>0$, then $c\neq 0$. Hence, $a>c$ or $c>0$ would imply that $c=0$ or $c\neq 0$ and therefore either that $\mathfrak{p}$ will not be tested after an even number of choices for a, or that it is impossible that $\mathfrak{p}$ will not be tested after an even number of choices for a. But as long as $\mathfrak{p}$ has not been tested, we can know nothing about the number of choices for a after which it will perhaps be tested. Although $a>0$, we cannot assert that $a>c$ or $c>0$, so that rule (6) of 7.3.1 is not fulfilled.

8.1.2. *Contradictoriness of classical real number arithmetic*

From here on I shall use Brouwer's terminology, which has now been sufficiently clarified to enable you to substitute another one if you prefer to do so.

The theorems of this section are stronger than those of the preceding one, in this respect, that they express the contradictority of some classical results.

Theorem 1. It is contradictory that for every real number a, $a\neq 0$ would imply $a\#0$.

Proof. Let J be a finitary spread which coincides with the interval [0, 1] (3.4.3, Th. 1). We define simultaneously an element

f of J and a real number-generator $a(f)=\{a_n(f)\}$ as follows. If $f \in J$, let $\mathfrak{p}(f)$ be the proposition "f is rational". For every n, f_n is chosen between the choice of $a_n(f)$ and that of $a_{n+1}(f)$. As long as $\mathfrak{p}(f)$ has not been tested, we choose $a_n(f)=2^{-n}$. If $\mathfrak{p}(f)$ is tested between the choice of a_m and that of a_{m+1}, we choose $a_{m+q}=2^{-m}$ for every q. As in the preceding example, $a(f) \neq 0$ for every f. Let us suppose that $a(f) \# 0$ for every f; then for every f a natural number $r(f)$ is known, so that $\mathfrak{p}(f)$ is tested before the choice of $a_{r+1}(f)$, that is before the choice of f_{r+1}. By the fan-theorem, a maximum s for $r(f)$ can be found, consequently $\mathfrak{p}(f)$ is tested for every f before the choice of f_{s+1}. Now let f be an ips which after the choice of f_s is subject to no other restrictions than those contained in the definition of J; we are free to continue f in such a way that either a rational or an irrational number results. Hence, it is contradictory that $\mathfrak{p}(f)$ is tested before the choice of f_{s+1}.

Thus $(\forall f)\,(a(f) \neq 0)$ and $\neg\,(\forall f)\,(a(f) \# 0)$ are both true, where f ranges over J.

FORM. Evidently, in this formal notation the quantifiers must be understood in an enlarged sense, corresponding with Brouwer's enlarged notion of an ips.

INT. The proof shows also that it is contradictory that, for every real number a, $a \neq 0$ and $a \not< 0$ would imply $a > 0$, or, in the terminology of 7.3.1, that $a \not> 0$ would imply $a > 0$.

In order to prove the contradictoriness of classical elementary arithmetic, Brouwer proves the following extension of Th. 1.

Theorem 2. It is contradictory, that for every real number a, $a \neq 0$ would imply $a \not> 0 \vee a \not< 0$.

Proof. As before, let J be a finitary spread coinciding with $[0,1]$, f any element of J, and $\mathfrak{p}(f)$ the proposition "f is rational". We define f and the real number-generator $b(f)=\{b_n(f)\}$ as follows. For every n, f_n is chosen between the choice of $b_n(f)$ and that of $b_{n+1}(f)$. As long as $\mathfrak{p}(f)$ has not been tested, we choose $b_n(f)=(-1)^n 2^{-n}$. If between the choice of $b_m(f)$ and that of $b_{m+1}(f)$ it is proved that f is irrational, we choose $b_{m+q}(f)=2^{-m}$ for every q. If between the choice of $b_m(f)$ and that of $b_{m+1}(f)$ it is proved that f cannot be irrational, we choose $b_{m+q}(f) = -2^{-m}$ for every q. As before, $b(f) \neq 0$. Suppose that for every f in J, either $b(f) \not> 0$ or $b(f) \not< 0$.

If f is irrational, then $b(f)>0$, so if $b(f) \not> 0$, then f cannot be irrational.

If f cannot be irrational, then $b(f)<0$, so if $b \not< 0$, then f is irrational. (Application of 7.1.2, (2) and (5)). Thus, [0, 1] would be split up into the subspecies of numbers which cannot be irrational and the subspecies of irrational numbers. This contradicts 3.4.5, Th. 2.

Corollary. It is contradictory that for every real number x either $x \not> 0$ or $x \not< 0$.

Theorem 3. It is contradictory that the equation $ax+b=0$, where $a \neq 0$ and $b \neq 0$, has always a solution.

Proof. Let c be a real number, $c \neq 0$. Take $a=c+2|c|$, $b=c-2|c|$, then $a \neq 0$ and $b \neq 0$. Let x_1 be a solution of $ax+b=0$. If $c>0$, then $a=-3b$, so $b(3x_1-1)=0$, so $x_1={}^1/_3$. If $c<0$, then $b=-3a$, so $a(x_1-3)=0$, so $x_1=3$. Thus, if $x_1 \neq {}^1/_3$, then $c \not> 0$, and if $x_1 \neq 3$, then $c \not< 0$. As either $x \neq {}^1/_3$ or $x \neq 3$, we have either $c \not> 0$ or $c \not< 0$. Hence, if $ax+b=0$ had a solution for every $c \neq 0$, then $c \neq 0$ would imply $c \not> 0$ or $c \not< 0$, which is contradictory by Th. 2.

Corollary. In euclidean plane geometry it is contradictory that every two lines which can neither coincide nor be parallel, intersect. For if $a \neq 0$ and $b \neq 0$, and if d is any real number $\neq 0$, then $ax+dy=b$ is the equation of a line which does not coincide with the axis of x and is not parallel to this axis. If all these lines intersected the axis of x, $ax=b$ would always have a solution if $a \neq 0$ and $b \neq 0$.

8.1.3. *Example concerning the Bolzano–Weierstrass theorem*

One of the forms of this theorem, discussed in 3.4.6, ran as follows:

(C) Every bounded species of real numbers without a point of accumulation is bounded in number.

Let $\mathfrak{p}$ be a proposition which has not been tested. I form an ips $\{b_n\}$ of rational numbers as follows. As long as $\mathfrak{p}$ has not been tested, I choose $b_n=2^{-n}$. If $\mathfrak{p}$ is tested between the choice of b_m and that of b_{m+1}, I choose $b_{m+p}=2^{-m}$ for every p. Let S be the species of the components b_n of this sequence. Suppose c were a point of

accumulation of S. It is clear that $c \not< 0$. But also $c \not> 0$, for from $c > 0$ it would follow that S were finite. Thus $c = 0$, but this implies that $\mathfrak{p}$ can never be tested, which is contradictory. Consequently, S can have no point of accumulation. On the other hand, if S were bounded in number, a natural number m would be known such that p will be tested before the choice of b_{m+1}, which is not the case. So there is no hope of proving (C) intuitionistically [L. E. J. Brouwer 1952B].

8.2. Negationless mathematics

Serious objections against the use of negation in mathematics have been raised by Griss [H. Freudenthal 1936A; A. Heyting 1936; G. F. C. Griss 1946, p. 24 and p. 64; 1948; 1948A]. Though agreeing completely with Brouwer's basic ideas on the nature of mathematics, he contends that every mathematical notion has its origin in a mathematical construction, which can actually be carried out; if the construction is impossible, then the notion cannot be clear.

Brouwer admits such theorems as "a square circle cannot exist"; we can prove this theorem by deriving a contradiction from the supposition that we had constructed a square which were at the same time a circle. According to Griss, this supposition has no clear sense, because it can never be realized. In other words, if a square circle does not exist, how can we have a clear notion of what it would be if it existed? Therefore Griss rejects negation as a mathematical concept.

In many cases the proof of a negative theorem suggests a positive form of it. For instance, in the proof of "no square circle exists" we meet some assertion like the following:

"If S is a square and P any point, then we can find points Q and R on the boundary of S, such that $PQ \# PR$." For Griss, this positive assertion expresses the real content of the negative theorem. Of course in most cases one negative theorem admits several positive translations. Griss tried to rebuild intuitionnistic mathematics without negation and reached some remarkable results in this direction [G. F. C. Griss 1946A; 1950; 1951].

In the arithmetic of integers and rational numbers negation is not essentially used. Here $a \neq b$ is the same as $a > b \vee a < b$, which

contains no negation. The first negative notion which occurred in these lectures was that of inequality between real numbers, $a \neq b$. Griss cannot admit this notion as well defined; he uses instead the relation $a \# b$, which is defined positively. However, among the basic properties of the relation $\#$ is the following: $\neg\, a \# b \to a = b$. Instead of this property Griss uses the following.

Theorem. If a and b are real numbers such that every real number c which lies apart from a lies also apart from b, then $a = b$.

It is not easy to give a negationless proof for this theorem; Griss succeeds only by an application of the fan-theorem.

Let a and b satisfy the hypothesis of the theorem. We may suppose that a and b are defined by canonical number-generators, $a = \{a_n 2^{-n}\}$, $b = \{b_n 2^{-n}\}$, such that $|a - a_n 2^{-n}| < 2^{-n}$, $|b - b_n 2^{-n}| < 2^{-n}$. For a given value of n, either $b_n < a_n$, $b_n = a_n$ or $b_n > a_n$. First suppose $b_n > a_n$. Consider the interval $i = [(a_n - 1)2^{-n},\ (b_n - 1)2^{-n}]$, and construct the finitary spread S_i which coincides with i, as in 3.4.3, Th. 1. Let $c = \{c_n 2^{-n}\}$ be an element of S_i, then $c < b$, so, by hypothesis, $c \# a$. This means that

$$(\exists k)\ (\exists m)\ (\forall p)\ (|c_{m+p} - a_{m+p}|2^{-m-p} > 2^{-k}).$$

Then, a fortiori,

$$(1) \qquad (\exists k)\ (|c_k - a_k|2^{-k} > 2^{-k}).$$

Thus to every element c of S there is associated a natural number k so that (1) is satisfied. By the fan-theorem we can find r so that k is known after that the first r components of c have been chosen; thus k takes but a finite number of values and we can find $k_0 = \max_{c \in S_i} k$, so that

$$(2) \qquad |c_{k_0} - a_{k_0}|2^{-k_0} > 2^{-k_0}$$

is valid for every c in S_i, that is, for every latticepoint $c_{k_0} 2^{-k_0}$ of order k_0 in i.

$a_{k_0} 2^{-k_0} > (a_n - 1)2^{-n}$; moreover, by (2), $a_{k_0} 2^{-k_0}$ is different from every latticepoint of order k_0 in i ("different" for rational numbers is a positive notion!), so $a_{k_0} 2^{-k_0} > (b_n - 1)2^{-n}$. But also $a_{k_0} 2^{-k_0} < (a_n + 1)2^{-n}$, so $b_n < a_n + 2$. We have supposed that $b_n > a_n$, so $b_n = a_n + 1$. Similarly, if $b_n < a_n$, then $b_n = a_n - 1$. We have proved that $|b_n - a_n| \leqq 1$ for every n, so $b = a$.

In the theory of spreads and species the null species cannot be defined. A species is defined by a property of mathematical objects, as explained in 3.2.1, but such a property can only have a clear sense after we have constructed an object which satisfies it. As a consequence of this exigency, the intersection of two species is only definable if it contains at least one element. For instance, we can speak of the species of algebraic fields, and of the species of species of six elements, but not of their intersection. The relation of difference between species is defined by induction as follows. Two species are different if one of them contains an element which is different from every element of the other one.

The logic of negationless mathematics is difficult to formalize, for several reasons. First of all, there is no calculus of propositions, because only true propositions make sense. Furthermore, the restriction on the use of conjunction between propositional functions (species) must be taken into account. Attempts at a formalization have been made by Griss [G. F. C. Griss 1949; 1950A], Gilmore [P. C. G. Gilmore 1953], and Vredenduin [P. G. J. Vredenduin 1953] and Valpola [V. Valpola 1955].

BIBLIOGRAPHY

This bibliography contains:
1. those writings to which reference is made in the text;
2. further important books and papers on intuitionism.

A nearly complete bibliography of intuitionism up to 1951 is to be found in [A. Heyting 1955].

ACZEL, P. H. G.

1966. "Saturated intuitionistic theories", Contributions to mathematical logic, Proc. Logic Coloq. Hannover 1966. (H. A. Schmidt, K. Schütte, H. J. Thiele, editors), Amsterdam 1968.

ASHVINIKUMAR

1966. Hilbert spaces in intuitionism. Thesis, Amsterdam.

1969. Über katalogisierte Räume. Compositio Math. **21**, p. 431–456.

1969A. The intuitionist contradictority of certain classical set-theoretic results. Nieuw Archief Wiskunde (3) **17**, p. 218–222.

1970 On the intuitionist theory of Stieltjes integration and its applications. Nederl. Akad. Wetensch. Proc. Ser. A **73** = Indag. math. **32**, p. 62–76.

1970A. On Brouwer–Stieltjes integration. Nederl. Akad. Wetensch. Proc. Ser. A 73 = Indag. Math. 32, p. 161–171.

A. Een ander bewijs van de intuïtionistische strijdigheid van het keuze-axioma. To appear.

BELINFANTE, M. J.

1931. Die Hardy–Littlewoodsche Umkehrung des Abelschen Stetigkeitssatzes in der intuitionistischen Mathematik. Proc. Akad. Amsterdam **34**, p. 401–412.

1941. Elemente der intuitionistischen Funktionentheorie. Proc. Akad. Amsterdam **44**, p. 173–185, 276–285, 420–425, 563–567, 711–717.

BETH, E. W.

1947. Semantical considerations on intuitionistic mathematics. Proc. Akad. Amsterdam **50**, p. 1246–1251 = Indagationes math. **9**, p. 572–577.

1948. Wijsbegeerte der wiskunde. 2nd ed. Antwerpen–Nijmegen 1948.

1955. Les fondements logiques des mathématiques. 2nd ed. Paris–Louvain 1955.

1956. Semantic construction of intuitionistic logic. Mededelingen Akad. Amsterdam N.R. **19**, no. 11.

1957. Remarks on elementary predicate logic. Nieuw Archief Wiskunde (3) **5**, p. 58–62.

1958. Construction sémantique de la logique intuitionniste. "Le raisonnement en mathématiques et en sciences expérimentales". Colloques internationaux du C.N.R.S. LXX. Paris 1958, p. 77–84.
1959. Remarks on intuitionistic logic. "Constructivity in mathematics", Proceedings of the colloquium held at Amsterdam, 1957. Amsterdam, 1959, p. 15–25.
1959A. Considérations heuristiques sur les méthodes de déduction par séquences. Logique et analyse 2, p. 153–159.
1960. Completeness results for formal systems. Proc. Intern. Congress Math. 1958, New York 1960, p. 281–288.
1962. Logique inférentielle et logique bivalente de l'implication. Bull. Soc. Math. Belge **14**, p. 120–132.

BOURBAKI, N.
1948. L'Architecture des mathématiques. Ed. Cahiers du Sud, 1948.
1949. Foundations of mathematics for the working mathematician. J. Symbolic Logic **14**, p. 1–8.

BROUWER, L. E. J.
1907. Over de grondslagen der wiskunde. Thesis. Amsterdam 1907.
1908. De onbetrouwbaarheid der logische principes. Tijdschrift voor wijsbegeerte **2**. (Reprinted in [L. E. J. Brouwer 1919]).
1912. Intuitionisme en formalisme. Groningen 1912. (English translation [L. E. J. Brouwer 1913].)
1913. Intuitionism and formalism. Bull. Amer. Math. Soc. **20**, p. 81–96.
1918. Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten. Erster Teil. Verhandelingen Akad. Amsterdam **12**, N° 5.
1919. Wiskunde, waarheid, werkelijkheid. Groningen 1919.
1919A. Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten. Zweiter Teil. Verhandelingen Akad. Amsterdam **12**, N° 7.
1919B. Intuitionistische Mengenlehre. Jahresbericht deutsch. Math. Ver. **28**, p. 203–208 = Proc. Akad. Amsterdam **23**, p. 949–954.
1920. Besitzt jede reelle Zahl eine Dezimalbruchentwicklung? Proc. Akad. Amsterdam **23**, p. 955–965.
1923. Begründung der Funktionenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten. Verhandelingen Akad. Wet. Amsterdam **13**, N° 2.
1924. Zur Begründung der intuitionistischen Mathematik I. Mathematische Annalen **93**, p. 244–258.
1924A. Über die Bedeutung des Satzes vom ausgeschlossenen Dritten in der Mathematik, insbesondere in der Funktionentheorie. J. reine angew. Math. (Crelle) **154**, p. 1–8.
1924B. Beweis, dass jede volle Funktion gleichmässig stetig ist. Proc. Akad. Amsterdam **27**, p. 189–194.
1924C. Intuitionistische Ergänzung des Fundamentalsatzes der Algebra. Proc. Akad. Amsterdam **27**, p. 631–634.

1924D. Bemerkungen zum Beweise der gleichmässigen Stetigkeit voller Funktionen. Proc. Akad. Amsterdam **27**, p. 644–646.

1925. Intuitionistische Zerlegung mathematischer Grundbegriffe. Jahresbericht deutsch. Math. Ver. **33**, p. 251–256.

1925A. Zur Begründung der intuitionistischen Mathematik II. Math. Annalen **95**, p. 453–473.

1925B. Intuitionistischer Beweis des Jordanschen Kurvensatzes. Proc. Akad. Amsterdam **28**, p. 503–506.

1926. Zur Begründung der intuitionistischen Mathematik III. Math. Annalen **96**, p. 451–489.

1926A. Über Definitionsbereiche von Funktionen. Math. Annalen **97**, p. 60–76.

1926B. Intuitionistische Einführung des Dimensionsbegriffes. Proc. Akad. Amsterdam **29**, p. 855–864.

1926C. Die intuitionistische Form des Heine–Borelschen Theorems. Proc. Akad. Amsterdam **29**, p. 866–868.

1927. Virtuelle Ordnung und unerweiterbare Ordnung. J. Reine angew. Math. (Crelle) **157**, p. 255–258.

1927A. Intuitionistische Betrachtungen über den Formalismus. Proc. Akad. Amsterdam **31**, p. 374–379 = Sitzungsber. preuss. Akad. Wiss. Berlin 1927, p. 48–52.

1927B. Beweis dass jede Menge in einer individualisierten Menge enthalten ist. Proc. Akad. Amsterdam **31**, p. 380–381.

1928. Die Struktur des Kontinuums. Vienna 1928.

1929. Mathematik, Wissenschaft und Sprache. Monatshefte Math. Phys. **36**, p. 153–164.

1933. Weten, Willen, Spreken. Euclides **9**, p. 177–193.

1942. Zum freien Werden von Mengen und Funktionen. Proc. Akad. Amsterdam **45**, p. 322–323 = Indagationes math. **4**, p. 107–108.

1942A. Die repräsentierende Menge der stetigen Funktionen des Einheitskontinuums. Proc. Akad. Amsterdam **45**, p. 443 = Indagationes math. **4**, p. 154.

1947. Richtlijnen der intuitionistische wiskunde. Proc. Akad. Amsterdam **50**, p. 339 = Indagationes math. **9**, p. 197.

1948. Consciousness, philosophy and mathematics. Proc. Xth intern. Congress Philosophy (Amsterdam 1948), p. 1235–1249.

1948A. Essentieel negatieve eigenschappen. Proc. Akad. Amsterdam **51**, p. 963–965 = Indagationes math. **10**, p. 322–324.

1948B. Opmerkingen over het beginsel van het uitgesloten derde en over negatieve asserties. Proc. Akad. Amsterdam **51**, p. 1239–1244 = Indagationes math. **10**, p. 383–388.

1949. De non-aequivalentie van de constructieve en de negatieve orderelatie in het continuum. Proc. Akad. Amsterdam **52**, p. 122–125 = Indagationes math. **11**, p. 37–40.

1949A. Contradictoriteit der elementaire meetkunde. Proc. Akad. Amsterdam **52**, p. 315–316 = Indagationes math. **11**, p. 89–90.

1950. Remarques sur la notion d'ordre. Comptes rendus Acad. Sci. Paris **230**, p. 263–265.

1950A. Sur la possibilité d'ordonner le continu. Comptes rendus Acad. Sci. Paris **230**, p. 349–350.

1951. On order in the continuum, and the relation of truth to non-contradictority. Proc. Akad. Amsterdam Ser. A **54**, p. 357–359 = Indagationes math. **13**, p. 357–359.

1952. Historical background, principles and methods of intuitionism. South-African J. Sci. **49**, p. 139–146.

1952A. An intuitionist correction of the fixed-point theorem on the sphere. Proc. Royal Soc. London A **213**, p. 1–2.

1952B. Over accumulatiekernen van oneindige kernsoorten. Proc. Akad. Amsterdam Ser. A **55**, p. 439–442 = Indagationes math. **14**, p. 439–442.

1952C. Door klassieke theorema's gesignaleerde pinkernen die onvindbaar zijn. Proc. Akad. Amsterdam Ser. A **55**, p. 443–445 = Indagationes math. **14**, p. 443–445.

1954. Points and spaces. Canadian J. math. **6**, p. 1–17.

1954A. Ordnungswechsel in Bezug auf eine coupierbare geschlossene stetige Kurve. Proc. Akad. Amsterdam Ser. A **57**, p. 112–114 = Indagationes math. **16**, p. 112–114.

1954B. Intuitionistische differentieerbaarheid. Proc. Akad. Amsterdam Ser A **57**, p. 201–204 = Indagationes math. **16**, p. 201–204.

1954C. An example of contradictority in classical theory of functions. Proc. Akad. Amsterdam Ser. A **57**, p. 204–206 = Indagationes math. **16**, p. 204–206.

1954D. Addenda en corrigenda over de rol van het principium tertii exclusi in de wiskunde. Proc. Akad. Amsterdam Ser. A **57**, p. 104–105 = Indagationes math. **16**, p. 104–105.

1954E. Nadere addenda en corrigenda over de rol van het principium tertii exclusi in de wiskunde. Proc. Akad. Amsterdam Ser. A **57**, p. 109–111 = Indagationes math. **16**, p. 109–111.

1955. The effect of intuitionism on classical algebra of logic. Proc. Roy. Irish Acad. Sect. A **57**, p. 107–112.

BROUWER, L. E. J. en B. DE LOOR.

1924. Intuitionistischer Beweis des Fundamentalsatzes der Algebra. Proc. Akad. Amsterdam **27**, p. 186–188.

BULL, R.A.

1965. A modal extension of intuitionistic logic. Notre Dame J. Formal Logic **6**, p. 142–146.

1966. MIPC as the formalisation of an intuitionist concept of modality. J. Symbolic Logic **31**, p. 609–616.

CARNAP, R.

1934. Logische Syntax der Sprache. Wien 1934.

1937. The logical syntax of language. London 1937.

CORPUT, J. G. VAN DER.
1946. On the fundamental theorem of algebra. Proc. Akad. Amsterdam **49**, p. 722–732, 878–886, 985–994 = Indagationes math. **8**, p. 430–440, 549–557, 605–614.

CURRY, H. B.
1951. Outlines of a formalist philosophy of mathematics. Amsterdam 1951.

DALEN, D. VAN
1963. Extension problems in intuitionistic plane projective geometry. Proc. Akad. Amsterdam Ser. A **66** p. 349–383 = Indagationes math. **25**, p. 349–383.
1968. A note on spread-cardinals. Compositio Math. **20**, p. 21–28.
1968A. Reducibilities in intuitionistic topology. J. Symbolic Logic **33**, p. 412–417.

DALEN, D. VAN AND A. S. TROELSTRA
1968. "Projections of lawless sequences". Intuitionism and proof theory, Proc. Summer Conf. Buffalo N.Y. 1968 (J. Myhill, ed.), Amsterdam-London 1970, p. 163–186.

DANTZIG, D. VAN.
1942. On the affirmative content of Peano's theorem on differential equations. Proc. Akad. Amsterdam **45**, p. 367–373 = Indagationes math. **4**, p. 140–146.
1942A. A remark and a problem concerning the intuitionistic form of Cantor's intersection theorem. Proc. Akad. Amsterdam **45**, p. 374–375 = Indagationes math. **4**, p. 147–148.
1947. On the principles of intuitionistic and affirmative mathematics. Proc. Akad. Amsterdam **50**, p. 918–929, 1092–1103 = Indagationes math. **9**, p. 429–440, 506–517.
1949. Comments on Brouwer's theorem on essentially-negative predicates. Proc. Akad. Amsterdam **52**, p. 949–957 = Indagationes math. **11**, p. 347–355.

DESTOUCHES, J.-L.
1951. Sur la mécanique classique et l'intuitionnisme. Proc. Akad. Amsterdam Ser. A **54**, p. 74–79 = Indagationes math. **13**, p. 74–79.

DIEUDONNÉ, J.
1949. L'axiomatique dans les mathématiques modernes. Congrès intern. de philosophie des sciences, Paris 1949. (Actualités scientifiques et industrielles 1137, Paris 1951.)

DIJKMAN, J. G.
1948. Recherche de la convergence négative dans les mathématiques intuitionnistes. Proc. Akad. Amsterdam **51**, p. 681–692 = Indagationes math. **10**, p. 232–243.
1952. Convergentie en divergentie in de intuitionistische wiskunde. Thesis, Amsterdam 1952.
1961. On Markov chains and intuitionism I. Proc. Akad. Amsterdam **64**, p. 314–327 = Indagationes math. **23**, p. 314–327.

1963. On Markov chains and intuitionism II. Proc. Akad. Amsterdam **66**, p. 275–281 = Indagationes math. **25**, p. 275–281.
1964. On Markov chains and intuitionism III. Proc. Akad. Amsterdam **67**, p. 256–261 = Indagationes math. **26**, p. 256–261.

DUBISLAV, W.
1932. Die Philosophie der Mathematik in der Gegenwart. Berlin 1932.

FITTING, M.
1968. "Intuitionistic model theory and the Cohen independence proofs". Intuitionism and proof theory, Proc. Summer Conf. Buffalo N.Y. 1968 (J. Myhill, ed.), Amsterdam-London 1970, p. 219–226.

FREUDENTHAL, H.
1936. Zum intuitionistische Raumbegriff. Compositio math. **4**, p. 82–111.
1936A. Zur intuitionistischen Deutung logischer Formeln. Compositio math. **4**, p. 112–116.

GHOSE, A.
1968. "Free choice sequences I, II". Automation in language, translation and theorem proving (P. Braffort and F. van Scheepen, ed.), Commission of the European Communities, Brussels 1968, p. 245–257, 259–261.

GIBSON, C. G.
1967. The Radon integral in intuitionism. Thesis, Amsterdam.
1969. The intuitionist measure, J. London Math. Soc. **44**, p. 617–624.

GILMORE, P.
1953. The effect of Griss' criticism of the intuitionistic logic on deductive theories formalized within the intuitionistic logic. Proc. Akad. Amsterdam Ser. A **56**, p. 162–187 = Indagationes math. **15**, p. 162–187.

GLIVENKO, V.
1928. Sur la logique de M. Brouwer. Bull. Acad. Sci. Belg. 1928, p. 225–228.
1929. Sur quelques points de la logique de M. Brouwer. Bull. Acad. Sci. Belg. 1929, p. 183–188.

GÖDEL, K.
1932. Zur intuitionistischen Arithmetik und Zahlentheorie. Ergebnisse eines math. Kolloquiums (Wien), Heft **4**, p. 34–38.
1932A. Eine Interpretation des intuitionistischen Aussagenkalküls. Ergebnisse eines math. Kolloquiums (Wien), Heft **4**, p. 39–40.
1958. Über eine bisher noch nicht benutzte Erweiterung des finiten Standpunktes. Dialectica **12**, p. 280–287.

GOODMAN, N. D.
1968. Intuitionistic arithmetic as a theory of constructions. Thesis, Stanford University.

1968A. "A theory of constructions equivalent to arithmetic". Intuitionism and proof theory, Proc. Summer Conf. Buffalo N.Y. 1968 (J. Myhill ed.), Amsterdam-London 1970, p. 121–150.

GOODSTEIN, R. L.

1956. A constructivist theory of plane curves. Fundamenta math. **43**, p. 23–35.

1954. Logic-free formalizations of recursive arithmetic. Math. Scand. **2**, p. 247–261.

1957. Recursive number theory. Amsterdam 1957.

1961. Recursive analysis. Amsterdam 1961.

GRISS, G. F. C.

1944. Negatieloze intuitionistische wiskunde. Verslagen Akad. Amsterdam **53**, p. 261–268.

1946. Idealistische Filosofie. Een humanistische levens- en wereldbeschouwing. Arnhem 1946.

1946A. Negationless intuitionistic mathematics I. Proc. Akad. Amsterdam **49**, p. 1127–1133 = Indagationes math. **8**, p. 675–681.

1948. Over de negatie. (Feestbundel, aangeboden aan H. J. Pos) Amsterdam 1948.

1948A. Mathématiques, mystique et philosophie. (Mélanges philosophiques. Bibliothèque du Xième Congrès intern. de philosophie) Amsterdam 1948.

1949. Logique des mathématiques intuitionnistes sans négation. Comptes rendus Acad. sci. Paris **227**, p. 946–947.

1950. Negationless intuitionistic mathematics II. Proc. Akad. Amsterdam **53**, p. 456–463 = Indagationes math. **12**, p. 108–115.

1950A. The logic of negationless intuitionistic mathematics. Proc. Akad. Amsterdam Ser. A, **54**, p. 41–49 = Indagationes math. **13**, p. 41–49.

1951. Negationless intuitionistic mathematics III, IV. Proc. Akad. Amsterdam **54** Ser. A, p. 193–200, 452–471 = Indagationes math. **13**, p. 193–200, 452–471.

1955. La mathématique intuitioniste sans négation. Nieuw Archief voor wiskunde (3) **3**, p. 134–142.

GRZEGORCZYK, A.

1964. A philosophically plausible formal interpretation of intuitionistic logic. Nederl. Akad. Wetensch. Proc. Ser. A. **67** = Indag. Math. **26**, p. 596–601.

HARROP, R.

1956. On disjunctions and existential statements in intuitionistic systems of logic. Math. Annalen **132**, p. 347–361.

1960. Concerning formulas of the types $A \to B \vee C$, $A \to (Ex)B(x)$ in intuitionistic formal systems. J. Symbolic Logic **25**, p. 27–32.

HENKIN, L.

1953. Some interconnections between modern algebra and mathematical logic. Trans. Amer. math. soc. **74**, p. 410–427.

HEYTING, A.

1925. Intuitionistische axiomatiek der projectieve meetkunde. Thesis Amsterdam. Groningen, 1925.

1927. Die Theorie der linearen Gleichungen in einer Zahlenspezies mit nichtkommutativer Multiplikation. Math. Annalen **98**, p. 465–490.

1927A. Zur intuitionistischen Axiomatik der projektiven Geometrie. Math. Annalen **98**, p. 491–538.

1929. De telbaarheidspraedicaten van Prof. Brouwer. Nieuw Archief voor wiskunde (2) **16**, p. 47–58.

1930. Die formalen Regeln der intuitionistischen Logik. Sitzungsber. preuss. Akad. Wiss. Berlin 1930, p. 42–56.

1930A. Die formalen Regeln der intuitionistischen Mathematik. Sitzungsber. preuss. Akad. Wiss. Berlin 1930, p. 57–71, 158–169.

1934. Mathematische Grundlagenforschung. Intuitionismus. Beweistheorie. (Ergebnisse der Math. und ihrer Grenzgebiete) Berlin 1934.

1935. Intuitionistische wiskunde. Mathematica B (Leiden) **4**, p. 72–82, 123–136; **5**, p. 62–80, 105–112; **7**, p. 129–141.

1936. Bemerkungen zu dem Aufsatz von Herrn Freudenthal: "Zur intuitionistischen Deutung logischer Formeln". Compositio math. **4**, 117–118.

1941. Untersuchungen über intuitionistische Algebra. Verhandelingen Akad. Amsterdam, 1. sectie **18** N° 2.

1946. On weakened quantification. J. Symbolic Logic **11**, 119–121.

1950. Espace de Hilbert et intuitionnisme. (Les méthodes formelles en axiomatique. Colloques internationaux du C.N.R.S. Paris 1953, p. 59–63).

1951. Note on the Riesz–Fischer theorem. Proc. Akad. Amsterdam Ser. A **54**, p. 35–40 = Indagationes math. **13**, p. 35–40.

1952. Logique et intuitionnisme. (Actes du 2e colloque intern. de logique math. Paris 1954, p. 75–82).

1953. Sur la théorie intuitionniste de la mesure. Bull. soc. math. Belg. **6**, p. 70–78.

1955. Les fondements des mathématiques. Intuitionnisme. Théorie de la démonstration. Paris–Louvain 1955.

1958. La théorie élémentaire de l'intégration en mathématiques intuitionnistes. "Le raisonnement en mathématiques et en sciences expérimentales", Colloques internationaux du C.N.R.S. LXX, Paris 1958.

1958A. Blick von der intuitionistischen Warte. Dialectica 12, p. 332–345.

1959. Some remarks on intuitionism. "Constructivity in mathematics", Proc. Colloquium Amsterdam 1957, Amsterdam 1959, p. 69–71.

1959A. Axioms for intuitionistic plane affine geometry. "The axiomatic method", Proc. intern. Symposion Berkeley 1957, Amsterdam 1959, p. 160–173.

1959B. Infinitistic methods from a finitist point of view. "Infinitistic methods" Proc. Symposion Warsaw 1959, Warszawa 1959, p. 185–192.

1960. Remarques sur le constructivisme. Logique et analyse 3, p. 177–182.

1961. Axiomatic method and intuitionism. Essays on the foundations of mathematics, p. 237–247. Jerusalem 1961.

1962. Méthodes et problèmes de l'intuitionnisme. Annales Faculté Sci. Clermont No 7, p. 101–106.

1962A. After thirty years. "Logic, Methodology and Philosophy of Science", Proc. 1960 intern. Congress, Stanford 1962, p. 194–197.

1964. "Remarques sur la théorie intuitionniste des espaces linéaires". E. W. Beth memorial colloquium, Paris 1964 (J.-L. Destouches, ed.), Dordrecht 1967.

1968. "Recent progress in intuitionistic analysis". Intuitionism and proof theory, Proc. Summer Conf. Buffalo N.Y. 1968 (J. Myhill, ed.) Amsterdam-London 1970, p. 95–100.

HILBERT, D.

1922. Neubegründung der Mathematik. Abhandl. mat. Seminar Hamburg. Univers. 1, p. 157–177.

HILBERT, D. und W. ACKERMANN.

1949. Grundzüge der theoretischen Logik. 3e Aufl. Berlin, Göttingen, Heidelberg 1949.

HOMAGK, F.

A. Intuitionistische Kennzeichnung der endlichen Spezies. To appear.

HOWARD, W. A.

1968. Functional interpretation of bar induction by bar recursion. Compositio Math. 20, p. 107–124.

HOWARD, W. A. AND G. KREISEL

1966. Transfinite induction and bar induction of types zero and one, and the role of continuity in intuitionistic analysis. J. Symbolic Logic 31, p. 325–358.

IONGH, J. J. DE.

1948. Restricted forms of intuitionistic mathematics. Proc. Xth intern. congress philosophy (Amsterdam 1948), p. 744–748.

JOHANSSON, I.

1936. Der Minimalkalkül, ein reduzierter intuitionistischer Formalismus. Compositio math. 4, p. 119–136.

JONGH, D. H. J. DE

1968. "A characterization of the intuitionistic propositional calculus". Intuitionism and proof theory, Proc. Summer Conf. Buffalo 1968 (J. Myhill, ed.), Amsterdam-London 1970, p. 211–218.

KLEENE, S. C.

1952. Introduction to metamathematics. Amsterdam–Groningen–New York 1952.

1952A. Recursive functions and intuitionistic mathematics. Proc. intern. Congr. Math. Cambridge (Mass.) 1950, Cambridge 1952, p. 679–685.

1962. Disjunction and existence under implication in elementary intuitionistic formalisms. J. Symbolic Logic **27**, p. 11–18. Addendum: ibid. **28**, p. 154–156.

1967. Constructive functions in "The foundations of intuitionistic mathematics". Logic, methodology and philosophy of science III (Proc. Third Intern. Congress, Amsterdam 1967; B. van Rootselaar and J. F. Staal, ed.), Amsterdam 1968, p. 137–144.

KLEENE, S. C. AND R. E. VESLEY.

1965. The foundation of intuitionistic mathematics. Amsterdam 1965.

KOLMOGOROFF, A.

1932. Zur Deutung der intuitionistischen Logik. Math. Zeitschrift 35, p. 58–65.

KREISEL, G.

1958. Elementary completeness properties of intuitionistic logic with a note on negations of prenex formulae. J. Symbolic Logic **23**, p. 317–330.

1958A. A remark on free choice sequences and the topological completeness proofs. J. Symbolic Logic **23**, p. 369–388.

1958B. Non-derivability of $\neg (x)A(x) \to (\exists x) \neg A(x)$, $A(x)$ primitive recursive, in intuitionistic formal systems. J. Symbolic Logic **23**, p. 456–457.

1959. Interpretation of analysis by means of constructive functionals of finite types. "Constructivity in mathematics", Proc. Colloquium Amsterdam 1957, Amsterdam 1959, p. 101–128.

1962. Foundations of intuitionistic logic. "Logic, Methodology and Philosophy of Science", Proc. 1960 intern. Congress, Stanford 1962, p. 198–210.

1962A. On weak completeness of intuitionistic predicate logic. J. Symbolic Logic **27**, p. 139–158.

1965. "Mathematical Logic". T. L. Saaty (ed.), Lectures on modern mathematics III, New York–London–Sydney 1965, p. 95–195.

1965A. "Informal rigour and completeness proofs". Problems in the philosophy of mathematics (Proc. Intern. Colloq. London 1965; I. Lakatos, ed.), Amsterdam 1967, p. 138–171.

1967. "Functions, ordinals, species". Logic, methodology and philosophy of science III (Proc. Third Intern. Congress, Amsterdam 1967; B. van Rootselaar and J. F. Staal, ed.), Amsterdam 1968, p. 145–159.

1968. Lawless sequences of natural numbers. Compositio Math. **20**, p. 222–248.

1968A. "Church's thesis: a kind of reducibility axiom for constructive mathematics". Intuitionism and proof theory, Proc. Summer Conf. Buffalo N.Y. 1968 (J. Myhill, ed.), Amsterdam–London 1970, p. 121–150.

KREISEL, G. AND H. PUTNAM

1957. Eine Unableitbarkeits-beweismethode für den intuitionistischen Aussagenkalkül. Archiv Math. Logik Grundlagenforsch. 3, p. 74–78.

KREISEL, G. AND A. S. TROELSTRA

1970. Formal systems for some branches of intuitionistic analysis. Annals math. logic 1, p. 229–387.

KRIPKE, S. A.

1963. Semantical considerations on modal and intuitionistic logic. Acta Philosophica Fennica 16, p. 83–94.

1963A. "Semantical analysis of intuitionistic logic I." Formal systems and recursive functions (Proc. Eighth Logic Colloq. Oxford 1963; J. N. Crossley and M. A. E. Dummett, ed.), Amsterdam 1965, p. 92–130.

KURODA, S.

1951. Intuitionistische Untersuchungen der formalistischen Logik. Nagoya math. J. 2, 35–47.

LÄUCHLI, H.

1968. "An abstract notion of realizability for which intuitionistic predicate calculus is complete". Intuitionism and proof theory, Proc. Summer Conf. Buffalo N.Y. 1968 (J. Myhill, ed.), Amsterdam–London 1970, p. 227–234.

LERDA, F.

1959. Analisi critica dei fondamenti dell' intuizionismo. Rendiconti Sem. mat. Univ. e Politec. Torino (1959–1960), p. 121–218.

LOOR, B. DE.

1925. Die hoofstelling van die algebra van intuisionistiese standpunt. Thesis. Amsterdam 1925.

MCKAY, C. G.

1967. A note on the Jaśkowski sequence. Z. Math. Logik Grundlagen Math. 13, p. 95–96.

MANNOURY, G.

1947. Les fondements psycholinguistiques des mathématiques. Neuchâtel 1947.

MEDVEDEV, Ju. T.

1966. Interpretation of logical formulas by means of finite problems. Dokl. Akad. Nauk SSSR 169, p. 20–23. Translation: Soviet Math. Dokl. 7, p. 857–860.

MENGER, K.

1930. Der Intuitionismus. Blätter deutsch. Philosophie 4, p. 311–325.

MOSCHOVAKIS, J. R.

1965. "Disjunction and existence in formalized intuitionistic analysis". Sets, models and recursion theory (Proc. Summer School Math. Logic and Tenth Logic Colloq. Leicester 1965; J. N. Crossley, ed.), Amsterdam 1967, p. 309–331.

MYHILL, J.

1966. Notes towards an axiomatization of intuitionistic analysis. Logique et Analyse (N.S.) 9, p. 280–297.

1967. "Formal systems of intuitionistic analysis I". Logic, methodology and philosophy of Science III (Proc. Third Intern. Congress, Amsterdam 1967; B. van Rootselaar and J. F. Staal, ed.), Amsterdam 1968, p. 161–178.

1968. "Formal systems of intuitionistic analysis II: the theory of species". Intuitionism and proof theory, Proc. Summer Conf. Buffalo N.Y. 1968 (J. Myhill, ed.), Amsterdam–London 1970, p. 151–162.

NEUMANN, J. VON.

1929. Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren. Math. Annalen 102, p. 49–131.

PIL'ČAK, B. JU.

1952. Ob isčislenii zadač (On the calculus of problems). Ukrainskiĭ mat. žurnal 4, p. 174–194.

PORTE, J.

1958. Une propriété du calcul propositionnel intuitionniste. Proc. Akad. Amsterdam Ser. A 61, p. 362–365 = Indagationes math. 20, p. 362–365.

RASIOWA, H. UND R. SIKORSKI.

1959. Formalisierte intuitionistische elementare Theorieen. "Constructivity in mathematics", Proc. Colloquium Amsterdam 1957, Amsterdam 1959, p. 241–249.

1963. The mathematics of metamathematics. Warszawa.

ROBINSON, ABRAHAM.

1951. On the metamathematics of algebra. Amsterdam 1951.

ROOTSELAAR, B. VAN.

1952. Un problème de M. Dijkman. Proc. Akad. Amsterdam Ser. A, 55, p. 405–407 = Indagationes math. 14, p. 405–407.

1954. Generalization of the Brouwer integral. Thesis, Amsterdam 1954.

1955. On the mapping of spreads. Proc. Akad. Amsterdam Ser. A, 58, p. 557–563 = Indagationes math. 17, p. 557–563.

1955A. Generating schemes for full mappings. Proc. Akad. Amsterdam Ser. A, 58, p. 646–649 = Indagationes math. 17, p. 646–649.

1956. A remark on Brouwermeasurable functions. Proc. Akad. Amsterdam Ser. A 59, p. 579–580 = Indagationes math. 18, p. 579–580.

1960. On intuitionistic difference relations. Proc. Akad. Amsterdam Ser. A 63, p. 316–322 = Indagationes math. 22, p. 316–322. Corrections: Proc. Akad. Amsterdam Ser. A 66, p. 132–133 = Indagationes math. 25, p. 132–133.

1964. On intuitionistic notions of divergence and convergence. Nieuw Arch. Wiskunde (3) 12, p. 19–24.

1968. "On subjective mathematical assertions". Intuitionism and proof theory, Proc. Summer Conf. Buffalo N.Y. 1968 (J. Myhill, ed.), Amsterdam–London 1970, p. 211–218.

ROSENBLOOM, P. C.

1954. Konstruktive Aequivalente für Sätze aus der klassischen Analysis. Proc. 2e Intern. Congress Internat. Union Philosophy of Science, Zurich 1954, II p. 135–137.

ROUSSEAU, G.

1968. Sheffer functions in intuitionistic logic. Z. Math. Logik Grundlagen Math. 14, p. 279–282.

SCHRÖTER, K.

1956. Über den Zusammenhang der in den Implikationsaxiomen vollständigen Axiomensysteme des zweiwertigen mit denen des intuitionistischen Aussagenkalküls. Zeitschr. Math. Logik Grundlagen Math. 2, p. 173–176.

1957. Eine Umformung des Heytingschen Axiomensystems für den intuitionistischen Aussagenkalkül. Zeitschr. Math. Logik Grundlagen Math. 3, p. 18–29.

SCHULTZ, K.

1966. Zur Darstellung der intuitionistischen Mathematik im Rahmen eines Beweisbarkeitskalküls. Monatsber. Deutsch. Akad. Wiss. Berlin 8, p. 399–402.

1967. Spreadtheorie und intuitionistische Topologie. Z. Math. Logik Grundlagen Math. 13, p. 251–264.

1967A. Speziestheorie und intuitionistische Topologie. Z. Math. Logik Grundlagen Math. 13, p. 377–378.

SCHÜTTE, K.

1962. Der Interpolationssatz der intuitionistischen Prädikatenlogik. Math. Annalen 148, p. 192–200.

1966. "Zur Semantik der intuitionistischen Aussagenlogik". Contributions to mathematical logic. Proc. Logic Colloq. Hannover 1966 (H. A. Schmidt, K. Schütte, H. J. Thiele, ed.), Amsterdam 1968, p. 231–235.

1968. Vollständige Systeme modaler und intuitionistischer Logik. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 42. Berlin–Heidelberg–New York 1968.

SCOTT, D.

1968. "Extending the topological interpretation to intuitionistic analysis". Intuitionism and proof theory, Proc. Summer Conf. Buffalo N.Y. 1968 (J. Myhill, ed.), Amsterdam–London 1970, p. 235–255.

SIKORSKI, R.

1959. Der Heytingsche Prädikatenkalkül und metrische Räume. "Constructivity in mathematics", Proc. Colloquium Amsterdam 1957, Amsterdam 1959, p. 250–253.

SKOLEM, TH.

1958. Remarks on the connection between intuitionistic logic and a certain class of lattices. Math. Scand. 6, p. 231–236.

SPECTOR, C.

1962. Provably recursive functionals of analysis: A consistency proof of analysis by an extension of principles formulated in current intuitionistic mathematics. "Recursive function theory", Proc. Symposia pure math. 5, p. 1–27.

STONE, M. H.

1932. Linear transformations in Hilbert space. New York 1932.

TAIT, W. W.

1967. Intensional interpretations of functionals of finite type I. J. Symbolic Logic 32, p. 198–212.

TARSKI, A.

1950. Some notions and methods on the borderline of algebra and metamathematics. Proc. intern. Congress Math. 1950, Cambridge, Mass., p. 705–720.

TEENSMA, E.

1954. The intuitionistic interpretation of analysis. Proc. 2e internat. Congress Internat. Union Philosophy of Science, Zurich 1954, II p. 144–150.

THOMASON, R. H.

1968. On the strong semantical completeness of the intuitionistic predicate calculus. J. Symbolic Logic 33, p. 1–7.

TROELSTRA, A. S.

1966. Intuitionistic general topology. Thesis, Amsterdam.

1966A. "The use of 'Brouwer's principle' in intuitionistic topology". Contributions to mathematical logic, Proc. Logic Colloq. Hannover 1966 (H. A. Schmidt, K. Schütte, H. J. Thiele, ed.), Amsterdam 1968, p. 289–298.

1967. Intuitionistic continuity. Nieuw Archief Wiskunde (3) 15, p. 2–6.

1967A. Finite and infinite in intuitionistic mathematics. Compositio Math. 18, p. 94–116.

1967B. "The theory of choice sequences". in: Proc. 3rd Intern. Congress Logic, Methodology and Philosophy of Science, Amsterdam 1967 (B. van Rootselaar and J. F. Staal, ed.), Amsterdam 1968, p. 201–223.

1968. New sets of postulates for intuitionistic topology. Compositio Math. 20, p. 211–221.

1968A. One-point compactifications of intuitionistic locally compact spaces. Fundamenta Math. 57, p. 75–93.

1969. Principles of intuitionism. Lecture notes in mathematics 95. Berlin–Heidelberg–New York 1969.

1969A. Notes on the intuitionistic theory of sequences I. Proc. Kon. Nederl. Akad. Wetensch. 72 = Indagationes math. 31, p. 430–440.

1970. Notes on the intuitionistic theory of sequences II, III. Proc. Kon. Nederl. Akad. Wetensch. 73 = Indagationes math. 32, p. 99–109, 245–252.

1970A. Informal theory of choice sequences. Studia Logica **25**, p. 31–54.

VALPOLA, V.

1955. Ein System der negationslosen Logik mit ausschliesslich realisierbaren Prädikaten. Acta philos. Fennica **9**, p. 1–247.

VESLEY, R. E.

1968. "A palatable substitute for Kripke's schema". Intuitionism and proof theory, Proc. Summer Conf. Buffalo N.Y. 1968 (J. Myhill, ed.), Amsterdam–London 1970, p. 211–218.

VOROB'EV, N. N.

1958. A new algorithm of derivability in the constructive propositional calculus. Trudy Mat. Inst. Steklov **52**, p. 193–225.

VREDENDUIN, P. G. J.

1953. The logic of negationless mathematics. Compositio math. **11**, p. 204–277.

WEYL, H.

1919. Über die neue Grundlagenkrise der Mathematik. Math. Zeitschr. **10**, p. 39–79.

1924. Randbemerkungen zu Hauptproblemen der Mathematik. Math. Zeitschr. **20**, p. 131–150.

INDEX

(Numbers refer to pages)

Absolute value, 27
Admissible sequence, 34
— latticepoint, 67
Almost everywhere, 75
Almost full, 75
Apart (number-generators), 19
— (in a field), 51
— (point), 60
— (in Hilbert space), 96
Apartness relation, see apart
Approximating strip, 76
Ascendant, 42
  immediate —, 34

Bar, 43
Barred, 43
Bolzano-Weierstrass theorem, 48, 123
Bounded in number, 49

Canonical number-generator, 41
— point-generator, 59
— p-g-spread, 60
Cauchy sequence, 16, 33
Characteristic determinant, 54
— function, 82
Closed interval, 40
— pointspecies, 61
Closure, 61
Closurepoint, 61
Coincide (number-generators), 16
— (real numbers with number-generators), 38
— (p-g-species), 60
— in the weak sense, 117
Complementary law, 35
Completion, 80
Component, 35
Congruent species, 39
—, geometrically, 60
Conjunction, 102
Continuation, 42
Continuum, 38
Contradiction, 102
Convergent, positively, 30
— (sequence of functions), 81
—, negatively, 113

Denumerably infinite, 39
Dependence, 56
  weak —, 56
  strong —, 56
Derived species, 61
Descendant, 42
  immediate —, 34
Detachable subspecies, 39
Disjunction, 102
Distance (between points), 60
— (between a point and a point-species), 66
Domain of a function, 78
  elementary —, 62
  rational elementary —, 62
  exterior elementary —, 62
Dually developable, 36

Element of a spread, 35
Elementary domain, 62
  rational —, 62
  exterior —, 62
— set of rectangles, 62
Equality
  (natural numbers), 15
  (elements of spreads), 36
  (spreads), 36
  (species), 38
Equivalent species, 39
— functions, 96

Exterior elementary domain, 62

Fan, 45
— theorem, 46
Field, 51
Finitary spread, 45
Finite, 39
Free vectors, 56
— (in Hilbert space), 96
Functions, characteristic, 82
—, rational, 22
—, bounded measurable, 76
—, measurable, 89

Geometrically congruent, 60

Heine-Borel theorem, 69

Identical number-generators, 16
Immediate ascendant, 34
— descendant, 34
Implication, 102
Inadmissible sequence, 34
— lattice-point, 67
Independent, 56
Inequality between number-generators, 17
Infinite, 40
  denumerably —, 39
Infinitely proceeding sequence, 32
Integral, 76, 91, 93
Interval (closed), 40
Inverse, 21
Ips, 32

Lattice-point, 67
Limit, 30
Limit-point, 61
Located, 66

Maximum, 26
Measure (elementary domain; region), 71
— (region-complement), 72
— (point-species), 83
Measurable region, 71
— region-complement, 72
— bounded function, 76
— pointspecies, 83
— function, 89
Member of a species, 37
Menge, 37
Metafunction, 96
Minimal calculus, 106
Minimum, 26

Naked spread, 37
Natural number, 13
Negatively convergent, 113
Negation, 102
Negligeable, 75
Non-negative part, 81
Non-oscillating, 113
Non-positive part, 75
Number
  natural —, 13
  real —, 38
  weak real —, 117
Number-generator, real, 16
—, canonical, 41
—, weak real, 117
Numerable, 40

Open, 62
Order relation, 25, 110
  partial — —, 110
  pseudo- — —, 110
  virtual — —, 111

Partial order relation, 110
Peano's axioms, 13
p–g, see point-generator
p–g-species, see point-generator species
p–g-spread, see point-generator spread
Point, 59
Point-generator, 59
  canonical —, 59
  — species, 59
  — spread, 60
  canonical — —, 60
Pointspecies, 59

Positively convergent, 30
Principal minor, 54
Pseudo-order relation, 110

Rank, 53
Rational elementary domain, 62
— function, 22
Real number, 38
  respectable — —, 27
Real number-generator, 16
Region, 63
  measurable —, 71
Region-complement, 63
  measurable —, 72
Relation, apartness —, see apart
Represent, 38
Respectable real number, 27
Riesz-Fischer theorem, 97

Segment, 42
Species, 37
  member of a —, 37
Split up, 39
Spread, 34
  finitary —, 45
  element of a —, 35
  naked —, 37
  — —, p–g, see point-generator-spread
Spread-law, 34
Spread-species, 38
Strong dependence, 56
Subspecies, 38
  detachable —, 39
Summable, 91, 93

Tested, 119
Type, 38

Unextensible, 112

Vector, 54, 92
Virtual order relation, 111

Weak dependence, 56
— real number-generator, 117
— real number, 117
Weakly greater, 118

GLOSSARY OF SYMBOLS

(Numbers refer to pages)

$\mathbf{1} \to n$	14	$\overset{*}{\cup}$	65
$>$, $<$	14, 25, 110	$\varrho(p, Q)$	66
$\neq$	17	$\varkappa_n$-square	73
$\#$	19, 51, 60	$f^+(x)$, $f^-(x)$	81
$a+b$, ab, $-a$, a^{-1}	21	$f_\varrho(x)$	82
$\not>$, $\not<$	25	$_Kf(x)$	90
$\lvert a \rvert$	27	$\approx$	96
Λ_M	34	$\wedge$, $\vee$	102
Γ_M	35	$\neg$	64, 102
$\in$, $\notin$, $\cup$, $\cap$	38	$\to$	102
$\subseteq$	38	$(\forall x)$, $(\exists x)$	106, 107
$[a, b]$	40	$\succ$, $\prec$	110
$\lvert x-y \rvert$	60	$^-\lim$	113
$R(V)$, $\alpha(V)$	62	$\supset$, $\vee$	116
$R^*(V)$, $\alpha^*(V)$	62	$\doteq$	117